Numerical Analysis Preliminary Exam

(Spring 2026)

First name: ______________________________ Student ID: ______________________________

Last Name: ______________________________ Additional pages: ________________________

Instructions:

1. All problems are worth 10 points.
2. Explain your answers clearly. Unclear answers will not receive credit.
3. State results and theorems you are using.
4. Use separate sheets for the solution of each problem.

PROBLEM 1. Consider the three data points

$$(0,1), \qquad (1,2), \qquad (2,2).$$

We want to fit these data by a line

$$p(x) = c_0 + c_1 x$$

in the least-squares sense.

(a) Write the least-squares problem in matrix form

$$\min_{c \in \mathbb{R}^2} \|Ac - b\|_2.$$

Identify A, c, and b.

(b) Let c be a least-squares solution, and let

$$r = b - Ac$$

be the residual. Explain why r must be orthogonal to the column space of A. Conclude that c satisfies the normal equations

$$A^T A c = A^T b.$$

(c) Solve the normal equations to find c_0 and c_1.

(d) In numerical software, least-squares problems are usually not solved by explicitly forming the normal equations. Why not? What is a more stable approach?

PROBLEM 2.

Consider a symmetric three-point quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(-a) + w_0 f(0) + w_1 f(a), \quad 0 < a < 1.$$

(a) Find a, w_0, w_1 so that the rule is exact for $1, x^2, x^4$.

(b) Show that the rule is automatically exact for all odd polynomials.

(c) Conclude that the rule is exact for all polynomials of degree at most 5.

(d) Show that the rule is not exact for x^6, and compute

$$\int_{-1}^{1} x^6\,dx - \left[w_1(-a)^6 + w_0 0^6 + w_1 a^6\right].$$

PROBLEM 3.

Let

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n}, \qquad m \geq n,$$

where $a_1, \ldots, a_n \in \mathbb{R}^m$.

(a) Write the classical Gram-Schmidt iteration for constructing orthonormal vectors $q_1, \ldots, q_n$ from the columns $a_1, \ldots, a_n$.

(b) Suppose the Gram-Schmidt process succeeds, and let

$$A = QR$$

be the resulting reduced QR factorization. In terms of the Gram-Schmidt iteration, what are the entries r_{ij} of R?

(c) Explain why R is upper triangular. Also explain why, if one of the diagonal entries r_{jj} is zero, then the columns of A are linearly dependent.

(d) Conversely, suppose the columns of A are linearly dependent. Explain why the Gram-Schmidt process must produce $r_{jj} = 0$ for some j.

PROBLEM 4.

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The normalized power method is defined by

$$y^{(k+1)} = Ax^{(k)}, \qquad x^{(k+1)} = \frac{y^{(k+1)}}{\|y^{(k+1)}\|_2}.$$

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) Use your answer to part (a) to find a formula for $A^k x^{(0)}$.

(c) Use your formula from part (b) to explain why the normalized power method converges to a multiple of the dominant eigenvector.

(d) Suppose

$$A^k x^{(0)} = \alpha_k v_1 + \beta_k v_2,$$

where v_1 is an eigenvector for the dominant eigenvalue and v_2 is an eigenvector for the other eigenvalue. Using your formula from part (d), compute

$$\frac{|\beta_{k+1}/\alpha_{k+1}|}{|\beta_k/\alpha_k|}.$$

What does this tell you about the rate at which the direction of $A^k x^{(0)}$ approaches the dominant eigenvector?

PROBLEM 5. Consider the one-parameter family of methods

$$y_{n+1} = y_n + h\left[(1-\theta)f(t_n, y_n) + \theta f(t_{n+1}, y_{n+1})\right], \qquad 0 \le \theta \le 1.$$

(a) Identify the methods corresponding to

$$\theta = 0, \qquad \theta = \frac{1}{2}, \qquad \theta = 1.$$

(b) Define the normalized local truncation error by

$$\tau_{n+1} = \frac{y(t_{n+1}) - y(t_n)}{h} - \left[(1-\theta)y'(t_n) + \theta y'(t_{n+1})\right].$$

Use Taylor expansion to determine how the order of consistency depends on θ.

(c) Apply the method to the test equation

$$y' = \lambda y, \qquad z = h\lambda.$$

Show that the stability function is

$$R(z) = \frac{1 + (1-\theta)z}{1 - \theta z}.$$

(d) Show that the absolute stability region is

$$\left\{ z \in \mathbb{C} : 2\,\mathrm{Re}(z) + (1-2\theta)|z|^2 \le 0 \right\}.$$

For which values of θ is the method A-stable?

PROBLEM 6. Consider the boundary value problem

$$-u''(x) = f(x), \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Let $h = 1/(n+1)$, $x_i = ih$, and let $u_i \approx u(x_i)$.

(a) Derive the finite difference system

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f(x_i), \qquad i = 1, \dots, n,$$

where $u_0 = u_{n+1} = 0$.

(b) Assuming $u \in C^4([0,1])$, show that the local truncation error is $O(h^2)$.

(c) Let

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Show that A is symmetric positive definite, and conclude that A is nonsingular.

(d) Give an $O(n)$ algorithm for solving the linear system $Au = b$. Explain why the operation count is $O(n)$.