Applied Mathematics Preliminary Exam
(Spring 2026)

Instructions:

1. **All problems are worth 10 points.**

2. **Explain your answers clearly. Unclear answers will not receive credit.**
   State results and theorems you are using.

3. **Use separate sheets for the solution of each problem.**

PROBLEM 1.

Consider the nonlinear system of ODEs

$$\frac{dx}{dt} = f(x;\mu)$$

in which $f(x;\mu)$ is odd symmetric in x, i.e.,

$$-f(x;\mu) = f(-x;\mu)$$

(a) Show that $x = 0$ is always a fixed point, and that all non-zero fixed points come in symmetric pairs.

(b) Explain why the fixed point at $x = 0$ cannot undergo a saddle-node bifurcation. [hint: Consider the Taylor series of $f(x;\mu)$ around the fixed point $x = 0$.]

(c) If a bifurcation of the fixed point at $x = 0$ does occur, what type of bifurcation is expected? Justify your answer.

(d) Could a saddle-node bifurcation occur at non-zero fixed points? Justify your answer.

PROBLEM 2.

Consider the system of differential equations

$$\frac{dx}{dt} = -xy$$
$$\frac{dy}{dt} = -x^2 - y$$

Determine the stability of the fixed point at $(0, 0)$.

PROBLEM 3.

Consider the integral operator

$$Ku(t) = \int_0^1 k(t,s)u(s)ds, \quad k(t,s) = \sum_{k=-2}^{2} k e^{i2\pi k(t-s)}.$$

1. Find the **nonzero** eigenvalues and the corresponding eigenfunctions of K.
2. Find the least squares solution to

$$Ku(t) = \cos(2\pi t) + a, \quad a \in (-\infty, \infty).$$

3. Find all values of a with which the least square solution in (b) becomes an exact solution.

PROBLEM 4.

Consider the operator

$$L_\lambda u(x) := u''(x) - \lambda u(x),$$

where $\lambda \neq 0$ is a real parameter.

1. Find the Green function $G(x,t)$ for $L_\lambda, x, t \geq 0$ with the initial conditions

$$G(0,t) = \partial_x G(0,t) = 0, \quad \forall t \geq 0.$$

2. Find the Green function $G(x,t)$ for $L_\lambda, x, t \in [0,1]$ with $\lambda > 0$ and the boundary conditions

$$G(0,t) = G(1,t) = 0, \quad \forall t \in [0,1].$$

PROBLEM 5.

Using any method, find the leading and first order behavior, as $x \to \infty$, of the integral

$$\int_0^{\pi^2/2} \int_0^{\pi^2/2} e^{x \cos(\sqrt{s+t})} \, ds \, dt.$$

PROBLEM 6.

This is a question on the WKB approximation. The wave equation in the presence of a single frequency wave, ω, in a spatially varying medium reduces to

$$\frac{d}{dx}\left[c^2(x)\frac{dh}{dx}\right] + \omega^2 h = 0, \tag{1}$$

where $c(x)$ is the prescribed wave speed and $h(x)$ is the height of the wave for which we must solve.

(a) What is the condition that the WKB approximation is a good approximation for this problem? Explain.

(b) Now, for an arbitrary $c(x)$ for which the WKB approximation is valid, perform the WKB approximation on equation (1), taking the theory out to third order, compute $h(x)$. (This will give the phase function, the amplitude functions, and the next correction.)

(c) Consider a medium for which $c(x) \to c_\pm$ as $x \to \pm\infty$ and $c(x) > 0$ for all x, and assume the WKB approximation remains valid for all x. What is the wavelength of the wave as $x \to \infty$? Use $h_\pm$ to denote the wave amplitude as $x \to \pm\infty$. Determine h_+ in terms of c_+, c_-, h_-.