Applied Math Prelim Examination (Spring 2015)

Instructions.

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State all results and theorems that you are using.
2. Use separate sheets for the solution of each problem.

1. Consider the following two sets of coupled ODEs.

Set 1 (Eqs. 1)

$$\begin{aligned} \frac{dx}{dt} &= -y - x(x^2+y^2) \\ \frac{dy}{dt} &= x - y(x^2+y^2) \end{aligned} \tag{1}$$

Set 2 (Eqs. 2)

$$\begin{aligned} \frac{dx}{dt} &= -y + xy^2 \\ \frac{dy}{dt} &= x - x^2y \end{aligned} \tag{2}$$

- Show that, for both sets of ODEs, linear stability predicts that the fixed point $(x=0, y=0)$ is a center.
- For one set of ODEs, the fixed point $(x=0, y=0)$ is, in fact, a stable spiral. Which one? Is it possible for the linearized equations to correctly predict the stability of the fixed-point? Why or why not?
- For one set of ODEs, the fixed point $(x=0, y=0)$ is, in fact, a center. Which one? Show that, for this set of ODEs, closed orbits exist.

2. Consider the following ODE

$$\frac{dx}{dt} = x(x-a)+b \tag{3}$$

- Sketch bifurcation diagrams for 1) $b=0$; 2) $b=\varepsilon$; and 3) $b=-\varepsilon$, where ε is a small, positive constant.

  (On your bifurcation diagram, indicate stable fixed points with a solid line, unstable fixed points with a dashed line and label all bifurcations).
- Sketch a stability diagram.

  (Recall that a stability diagram will have a and b as axes, and will indicate regions where there are different numbers of fixed points).

3. Consider waves in a resistant medium that satisfy the problem

$$\begin{aligned} u_{tt} =& u_{xx} - \mu u_t, \text{ for } 0 < x < \pi \\ u_x(0,t) = 0, &\quad u_x(\pi,t) + u(\pi,t) = 0, \\ u(x,0) =& \phi(x), \quad u_t(x,0) = \psi(x), \end{aligned}$$

where $\mu > 0$ is a constant. Write down the Fourier series expansion of the solution.

4.

(a) Show that

$$\int_a^x \int_a^s f(t)dtds = \int_a^x (x-t)f(t)dt.$$

(b) Express the linear second order ODE,

$$\begin{aligned} y'' + \alpha y' + c^2 y = 0. \\ y(0) = 0, \ y'(0) = 1, \end{aligned}$$

as an integral equation of the form

$$y(x) = h(x) + \int_0^x K(x,t)y(t)dt.$$

Determine the functions $h(x)$ and $K(x,t)$?

(c) What is the asymptotic behavior of y (as $x \to \infty$) as a function of the sign of α?

5. Find the the first two terms in the asymptotic approximation of the integral

$$\int_0^1 e^{x[t(1-t^2)]}\, dt.$$

in the following two limits: (a) $x \to -\infty$ and (b) $x \to \infty$. (Hint. Use two different methods to study the cases (a) and (b).)

6. The equation of motion for a pendulum of length L is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0\,,$$

where $\theta(t)$ is the angle measured from the downward vertical direction, and g is the acceleration of gravity. For small initial data,

$$\theta(0) = \epsilon \ll 1, \quad \frac{d\theta}{dt}(0) = 0$$

use the method of multiple scales to calculate the first two terms in the asymptotic expansion (in ϵ) of the frequency of the pendulum. You will need to introduce the slow time scale $T = \epsilon^2 t$.