Applied Mathematics Preliminary Exam
(Spring 2021)

Instructions:
1. All problems are worth 10 points.
2. Explain your answers clearly. Unclear answers will not receive credit.
State results and theorems you are using.
3. Use separate sheets for the solution of each problem.

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PROBLEM 1. Compare and contrast second order Hamiltonian and gradient systems. Let

$$F(x,y)=\frac{y^2}{2}+1-\cos^2(x), \quad -\pi\le x<\pi$$

(a) Consider a gradient system whose potential is given by $\vec{x}=(x,y),\ V(\vec{x})=F(x,y)$ and

$$\frac{d\vec{x}}{dt}=-\nabla V$$

Find the fixed points and analyze their linear stability. Plot contours of constant V and sketch some trajectories on this plot. Explicitly find the fixed points, and compute their stable and unstable manifolds.

(b) Show that there can be no oscillations in any gradient flow.

(c) Consider a Hamiltonian system whose Hamiltonian is given by $H(x,y)=F(x,y)$.

$$\frac{dx}{dt}=\frac{\partial H}{\partial y}, \quad \frac{dy}{dt}=-\frac{\partial H}{\partial x}.$$

Find the fixed points and analyze their linear stability. Plot contours of constant H and sketch some trajectories on this plot.

(d) Find an approximation for the period of orbits near the homoclinic orbit, i.e. $T(H)$ for H near 0.

(e) Identify the homoclinic trajectory. What is the value of H on the homoclinic trajectory? Call this H_*. How much time does it take to traverse the homoclinic orbit?

PROBLEM 2. The Brusselator is a simple model of a hypothetical chemical oscillator. In dimensionless form, its kinetics are

$$\begin{aligned}\dot{x} &= 1-(b+1)\,x+a\,x^2\,y\\ \dot{y} &= b\,x-a\,x^2\,y\end{aligned}$$

where $a,\ b>0$ and $x,\ y>0$ are dimensionless chemical concentrations.

(a) Sketch the nullclines and find all of the fixed points. Use the Jacobian to classify them.

(b) Construct and draw a trapping region for the flow (think Poincare-Bendixson here). Hint: if you add the equations you get an equation for $d(x+y)/dt$.

(c) Show that a Hopf bifurcations occurs at some parameter value $b=b_c$ and determine this value.

(d) Does the limit cycle exist for $b>b_c$ or $b<b_c$? Explain, using the Poincare-Bendixson theorem.

(e) Find the approximate period of the limit cycle for b very near b_c.

PROBLEM 3. Let

$$Au=(1-x^2)u''-2xu', \quad x\in(-1,1).$$

(a) Find the minimum principle whose stationary point u satisfies $Au=0$ with the boundary condition $u(-1)=0, u(1)=1$. Show explicitly that the first variation of your minimum principle vanishes with such u.

(b) Find Green's function of A with the boundary condition $|u(\pm1)|<\infty$.

PROBLEM 4. Consider the Laplacian Δ in $D=[0,a]\times[0,b]$ with the homogeneous Dirichlet boundary condition on the boundary ∂D.

(a) Find all the eigenvalues and eigenfunctions.

(b) Find Green's function.

(c) Find the solution to the Dirichlet problem: $\Delta u(x,y)=0$ in D with the boundary condition

$$u|_{\partial D}=x^2|_{\partial D},$$

i.e. the function x^2 restricted to ∂D.

PROBLEM 5. Use the method of multiple scales to find the leading order expansion of the solution that is valid for large time in the limit of small ϵ to

$$\ddot{u}+\epsilon\,(1+\gamma\cos(u))\,\dot{u}+\sin(u)=\epsilon a, \quad u(0)=0,\ \dot{u}(0)=0.$$

where $\gamma>0$ and a are constants.

PROBLEM 6. Consider the integral

$$I(x)=\int_0^1 \exp(x\ t(t-t^2))\ dt.$$

(i) Find the leading term in the asymptotic approximation to $I(x)$ as $x\to-\infty$.
(ii) Extend your work to find the first two-terms in the asymptotic approximation to $I(x)$ as $x\to-\infty$.