Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 25, 2007

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f\left(x^n\right) \mathrm{d}x$$

exists and evaluate the limit. Does the limit always exist if f is only assumed to be integrable?

Problem 2: (10 points)

Suppose that for each $n \in \mathbb{Z}$, we are given a real number ω_n. For each $t \in \mathbb{R}$, define a linear operator $T(t)$ on 2π-periodic functions by

$$T(t)\left(\sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{inx}\right) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega_n t} f_n \mathrm{e}^{inx},$$

where $f(x) = \sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}nx}$ with $f_n \in \mathbb{C}$.

(a) Show that $T(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a unitary map.

(b) Show that $T(s)T(t) = T(s+t)$ for all $s, t \in \mathbb{R}$.

(c) Prove that if $f \in C^\infty(\mathbb{T})$, meaning that it has continuous derivatives of all orders, then $T(t)f \in C^\infty(\mathbb{T})$.

Problem 3: (10 points)

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces, with X compactly imbedded in Y, and Y continuously imbedded in Z (meaning that: $X \subset Y \subset Z$; bounded sets in $(X, \|\cdot\|_X)$ are precompact in $(Y, \|\cdot\|_Y)$; and there is a constant M such that $\|x\|_Z \leq M\|x\|_Y$ for every $x \in Y$). Prove that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$\|x\|_Y \leq \varepsilon\|x\|_X + C(\varepsilon)\|x\|_Z \qquad \text{for every } x \in X.$$

Problem 4: (10 points)

Let $\mathcal{H}$ be the weighted L^2-space

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} e^{-|x|} |f(x)|^2 \, dx < \infty \right\}$$

with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} g(x) \, dx.$$

Let $T : \mathcal{H} \to \mathcal{H}$ be the translation operator

$$(Tf)(x) = f(x+1).$$

Compute the adjoint T^* and the operator norm $\|T\|$.

Problem 5: (10 points)

(a) State the Rellich Compactness Theorem for the space $W^{1,p}(\Omega)$ for $\Omega \subset \mathbb{R}^n$. Recall that the Sobolev conjugate exponent is defined as $p^* = \frac{np}{n-p}$, and that there are some constraints on the set Ω.

(b) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^3$ open, bounded, and smooth. Show that there exists an $f \in H^1(\Omega)$ such that for a subsequence $\{f_{n_\ell}\}_{\ell=1}^{\infty}$,

$$f_{n_\ell} D f_{n_\ell} \rightharpoonup f Df \quad \text{weakly in } L^2(\Omega),$$

where $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$ denotes the (weak) gradient operator.

Problem 6: (10 points)

Let $\Omega := B\left(0, \frac{1}{2}\right) \subset \mathbb{R}^2$ denote the open ball of radius $\frac{1}{2}$. For $x = (x_1, x_2) \in \Omega$, let

$$u(x_1, x_2) = x_1 x_2 \left[\log\left(|\log(|x|)| \right) - \log\log 2 \right] \quad \text{where } |x| = \sqrt{x_1^2 + x_2^2}.$$

(a) Show that $u \in C^1(\bar{\Omega})$.

(b) Show that $\frac{\partial^2 u}{\partial x_j^2} \in C(\bar{\Omega})$ for $j = 1,2$, but that $u \notin C^2(\bar{\Omega})$.

(c) Using the elliptic regularity theorem for the Dirichlet problem on the disc, show that $u \in H^2(\Omega)$.

Problem 7: (8 points)

Write down the general solution for the following two linear dynamical systems. Draw the phase plane with trajectories, and eigenvectors clearly labeled. Compare the two systems specifically paying attention to the nature of their eigenvectors and eigenvalues. Explicitly write the solution to both systems for the initial condition $x(0) = 0,\ y(0) = 1$. Sketch the plot of $x(t)$ on the same axes for each of these solutions.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Problem 8: (12 points)

Consider the second order dynamical system

$$\dot{x} = x\left[x(1-x) - y\right]$$
$$\dot{y} = y(x-a)$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and a is a positive control parameter.

(a) Sketch the nullclines in the positive quadrant of the x, y plane.

(b) Find the fixed points and classify their (linearized) stability.

(c) Sketch the phase portrait for $a > 1$. What happens to the predators in this case?

(d) Find the value of a for which a Hopf bifurcation occurs.

(e) Estimate the frequency of the limit cycle oscillations for a near the bifurcation.

(f) Sketch all the topologically distinct phase portraits for $0 < a < 1$.