Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
March 24, 2011

Instructions:

- *This exam has 4 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Let $\Omega = (0,1)$, the open unit interval in $\mathbb{R}$, and consider the sequence of functions $f_n(x) = n\mathrm{e}^{-nx}$. Prove that $f_n \not\rightharpoonup f$ weakly in $L^1(\Omega)$, i.e., the sequence f_n does not converge in the weak topology of $L^1(\Omega)$.
(**Hint**: Prove by contradiction.)

Problem 2: (10 points)

Let $\Omega = (0,1)$, and consider the linear operator $A = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ acting on the Sobolev space of functions X where

$$X = \left\{ u \in H^2(\Omega) \mid u(0) = 0, u(1) = 0 \right\},$$

and where

$$H^2(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; \frac{\mathrm{d}u}{\mathrm{d}x} \in L^2(\Omega), \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \in L^2(\Omega) \right\}.$$

Find all of the eigenfunctions of A belonging to the linear span of

$$\{\cos(\alpha x), \sin(\alpha x) \mid \alpha \in \mathbb{R}\},$$

as well as their corresponding eigenvalues.

Problem 3: (10 points)

Let $\Omega = (0,1)$, the open unit interval in $\mathbb{R}$, and set

$$v(x) = (1 + |\log x|)^{-1}.$$

Show that $v \in W^{1,1}(\Omega)$ and that $v(0) = 0$, but that $\dfrac{v}{x} \notin L^1(\Omega)$. (This shows the failure of Hardy's inequality in L^1.) Note that $W^{1,1}(\Omega) = \left\{ u \in L^1(\Omega) \;\middle|\; \dfrac{\mathrm{d}u}{\mathrm{d}x} \in L^1(\Omega) \right\}$, where $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ denotes the weak derivative.

Problem 4: (10 points)

Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Show that

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} b_n \tau_n \delta \text{ in } \mathscr{D}', \tag{1}$$

that is, that equality in equation (1) holds in the sense of distributions, and relate b_n to the coefficients of the Fourier series. Note that δ denotes the Dirac distribution and τ_y is the translation operator, given by $\tau_y f(x) = f(x+y)$.

(**Hint:** Write $f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}$ with convergence in $L^2(0,2\pi)$ and where the coefficients $c_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nx} f(x)\,\mathrm{d}x$.)

Problem 5: (10 points)

Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Given $\epsilon > 0$, prove that for $N < \infty$ there is a finite Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)] \tag{2}$$

such that

$$|\phi(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R}.$$

This shows that the space of real-valued trigonometric polynomials on $\mathbb{R}$ (functions which can be expressed as in (2)) are *uniformly* dense in the space of periodic continuous function on $\mathbb{R}$ with period 2π.

(**Hint:** The Stone-Weierstrass theorem states that if X is compact in $\mathbb{R}^d$, $d \in \mathbb{N}$, then the algebra of all real-valued polynomials on X (with coordinates $(x_1, x_2, \ldots, x_d)$) is dense in $C(X)$.)

Problem 6: (10 points)

For $\alpha \in (0,1]$, the space of Hölder continuous functions on the interval $[0,1]$ is defined as

$$C^{0,\alpha}([0,1]) = \{u \in C([0,1]) \; : \; |u(x) - u(y)| \le C|x-y|^{\alpha}, x, y \in [0,1]\},$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x\in[0,1]} |u(x)| + \sup_{x,y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}.$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) \; : \; \|u\|_{C^{0,\alpha}([0,1])} \le 1\}$ is a compact set in $C([0,1])$.

(**Hint:** The Arzela-Ascoli theorem states that if a family of continuous functions U is equicontinuous and uniformly bounded on $[0,1]$, then each sequence u_n in U has a uniformly convergent subsequence. Recall that U is uniformly bounded on $[0,1]$ if there exists $M > 0$ such that $|u(x)| < M$ for all $x \in [0,1]$ and all $u \in U$. Further, recall that U is equicontinuous at $x \in [0,1]$ if given any $\epsilon > 0$, there exists $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ for all $|x - y| < \delta$ and every $u \in U$.)

Problem 7: (10 points)

Consider the system of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x(y+1)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - x^2 - y^2$$

(a) Show that $(x, y) = (0, 1)$ and $(0, -1)$ are fixed point of the system. Linearize the system about the fixed points $(0, 1)$ and $(0, -1)$ and use linearized system to classify the fixed points.

(b) Sketch the phase portrait of the full system and re-classify the fixed points.

Problem 8: (10 points)

Consider the system describing a particle mass moving in a double-well potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, i.e.,

$$\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x} = x - x^3.$$

(a) Show that the energy $E(x, \dot{x}) = \frac{\dot{x}^2}{2} + V(x)$ is a conserved quantity for this system, i.e. $E(x, \dot{x})$ is constant along trajectories.

(b) Sketch the $x, \dot{x}$-phase portrait. Classify the fixed points of the system $(0, 0)$ and $(\pm 1, 0)$.