Fall 2012: PhD Applied Math Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1. Consider the one-dimensional discrete dynamical system

$$x_{n+1} = \mu e^{x_n}, \qquad n = 0, 1, 2, \ldots$$

where $x_n \in \mathbb{R}$ and μ is a real parameter.

(a) Describe qualitatively how the the fixed points of the system change as μ increases from $-\infty$ to ∞ and determine their stability.

(b) What types of bifurcation occur when there is a change in stability of the fixed points?

Problem 2. Consider the 3×3 system of ODEs

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1, \qquad \dot{x}_3 = 1 - \left(x_1^2 + x_2^2\right)$$

(a) Show that trajectories of the system in phase space $\{(x_1, x_2, x_3) \in \mathbb{R}^3\}$ lie on the cylinders

$$x_1^2 + x_2^2 = c^2 \tag{1}$$

where $c \geq 0$ is a constant.

(b) Sketch the trajectories on the cylinder (1) for: (i) $c = 0$; (ii) $0 < c < 1$; (iii) $c = 1$; (iv) $c > 1$.

(c) Does the system have any equilibria? Does it have periodic solutions? Why doesn't your answer contradict the Poincaré- Bendixson theorem?

Problem 3. Compute the Green's function for the boundary value problem

$$\begin{aligned} &u'' + u = f(x) \qquad && 0 < x < 1, \\ &u'(0) = 0, \qquad u(1) = 0, && \end{aligned}$$

and write out the Green's function representation of the solution.

Problem 4. Suppose $u(x)$ satisfies the following boundary value problem on $[-1, 1]$

$$\begin{aligned} Lu &= f(x) \qquad -1 < x < 1, \\ u(-1) &= 0, \qquad u(1) = 0. \end{aligned} \tag{2}$$

where $f : [-1, 1] \to \mathbb{R}$ is a given smooth function and

$$Lu = u'' + xu' + 3u.$$

(a) Find the formal adjoint L^* of L and the adjoint boundary conditions.
(b) Verify that $v(x) = 1-x^2$ is a solution of the homogeneous adjoint problem and derive a necessary condition that $f(x)$ must satisfy if (2) is solvable.

Problem 5. (a) Suppose that $0 < \epsilon \ll 1$ is a small positive parameter. Use the method of matched asymptotic expansions to construct leading order approximations

$$x = x_0 + O(\epsilon), \qquad y = y_0 + O(\epsilon) \qquad \text{as } \epsilon \to 0^+$$

of the solution $x(t; \epsilon)$, $y(t; \epsilon)$ of the initial value problem

$$\dot{x} = -xy, \qquad \epsilon\dot{y} = x^2 - y, \qquad x(0) = x_0, \quad y(0) = y_0$$

that are valid for times of the order ϵ and times of the order 1.
(b) Sketch the phase plane of this system. How do solutions behave as $t \to +\infty$?

Problem 6. (a) Find all separable solutions $u(x,t) = F(x)G(t)$ of the wave equation on the interval $0 < x < 1$ subject to homogeneous Dirichlet boundary conditions:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad 0 < x < 1 \\ u(0,t) &= 0, \qquad u(1,t) = 0. \end{aligned}$$

(b) Consider a Dirichlet problem for the wave equation on a rectangle of sides length 1 and $T > 0$,

$$\begin{array}{lll} u_{tt} - u_{xx} = 0 & & 0 < x < 1, \quad 0 < t < T, \\ u(x,0) = f(x), & u(x,T) = g(x) & 0 \le x \le 1, \\ u(0,t) = h(t), & u(1,t) = k(t) & 0 \le t \le T, \end{array}$$

where $f, g : [0,1] \to \mathbb{R}$ and $h, k : [0,T] \to \mathbb{R}$ are given functions. Suppose that this problem has a solution. Show that solutions are unique if T is irrational and non-unique if T is rational.