Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(3 January 2002)

Problem 1. (5 points) Let V be a metric space with the property that every sequence $(x_k)_{k\geq 1}$ such that

$$d\left(x^k, x^l\right) < 3^{-k} \qquad \text{for } l \geq k \geq 1$$

is convergent. Prove that V is complete.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded function, and define

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) e^{-y^2/2}\, dy$$

where $-\infty < x < \infty$ and $t \geq 0$.

a) (5 points) Show that $u(x,t)$ is a solution of the initial value problem

$$\begin{aligned} u_t &= u_{xx} - xu_x && -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x) && -\infty < x < \infty. \end{aligned}$$

b) (5 points) What is the asymptotic behavior of $u(x,t)$ as $t \to \infty$?

Problem 3. Let $\mathcal{H}$ denote a Hilbert space with inner product $(\cdot,\cdot)$. For any two vectors $f, g \in \mathcal{H}$, define the operator $f \otimes g$ by

$$(f \otimes g)\, v = f\ (g, v)\, , v \in \mathcal{H}$$

Let $\{\varphi_k\}_{k=0,1,2,\ldots}$ be an orthornormal basis for $\mathcal{H}$. For each positive integer N define the operator K_N by

$$K_N = \sum_{k=0}^{N-1} \varphi_k \otimes \varphi_k$$

a) (5 points) What is the dimension of the range of K_N?

b) (5 points) Prove that K_N is a projection operator.

Problem 4. Let $\{f_n\}$ denote a sequence of vectors in the Hilbert space $\mathcal{H}$.

a) (5 points) Define the notion of *strong convergence* of this sequence to a vector $f \in \mathcal{H}$.

b) (5 points) Define the notion of *weak convergence* of this sequence to a vector $f \in \mathcal{H}$.

c) (5 points) Give an example *to show* that a sequence can converge weakly but not strongly. Be sure to show that your sequence converges weakly but does not converge strongly.

d) (5 points) Let A be a bounded operator on $\mathcal{H}$ and $\{f_n\}$ a sequence of vectors that converge strongly to f. Prove that $\{Af_n\}$ converges strongly to Af.

Problem 5. Let A denote a bounded operator on the Hilbert space $\mathcal{H}$.

a) (5 points) Define the adjoint operator A^*.

b) (5 points) What does it mean for the operator A to be self-adjoint?

c) (5 points) Prove that if λ is any eigenvalue of a self-adjoint operator A, then λ is a real number.

d) (5 points) Let A be self-adjoint and λ a complex number with nonzero imaginary part. Define the resolvent operator R_λ. What can you say about R_λ?

Problem 6. Consider a non-linear autonomous system of ODE's

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad .$$

a) (5 points) Define a *rest point* of this system, and explain in what sense it corresponds to a solution.

b) (5 points) Give the precise definition of *stable* and *asymptotically stable* rest point.

Problem 7. The orbit of a planet in general relativity follows a trajectory of the ODE

$$\ddot{u} = -P'(u) \quad ,$$

where $u = r^{-1}$, $r =$ distance from the planet to the sun (assumed to be point masses), and

$$P(u) = -Cu(u - u_2)(u - u_3) \quad .$$

Here $C > 0$ is a constant, and $0 < u_2 < u_3 < \infty$, are critical values of u.

a) (5 points) Define the energy and sketch the phase portrait, noting the character of all rest points.

b) (5 points) Show which values of the energy correspond to bounded periodic orbits.

c) (5 points) Use the phase portrait to argue that any orbit sufficiently close to the sun, with sufficiently small energy, will fall into the sun (evidence of a black hole).

Problem 8. Consider the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \quad ,$$

where

$$A = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix} \quad .$$

a) (5 points) Find $\exp A$, and use it to give a formula for the solution set.

b) (5 points) Prove that

$$\lim_{t \to +\infty} \mathbf{x}(t) = 0$$

for any solution $\mathbf{x}(t)$.