This collection of 12 problems is intended to help you review and prepare for the prelim on *Probability*. The exam tests for knowledge of MAT 135A (Probability) and MAT 135B (Stochastic Processes). Try to write the solutions as you would in the exam: Write all details when solving every problem. Be organized and use the notation appropriately. Initially try to solve the problems without any assistance.

About the exam:

- The prelim exam will be 3 hours long and have 8 problems.
- The exam is "closed book": no calculators/books/notes/etc allowed.
- An equation sheet will be provided (see below).
- You must show your work and fully justify all answers.
- "Combinatorial" answers should not be simplified, but left in a descriptive form that shows how you arrived at the answer. This may include Binomial coefficients, factorials, etc.
- Closed-form answers — not containing an integral or summation sign — should be given unless stated otherwise.
- Leave answers in terms of the standard Normal c.d.f. Φ when appropriate.

Equations:

- Poisson(λ) — mean λ; variance λ; p.m.f. $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$
- Geometric(p) — mean $1/p$; possible values $k = 1, 2, 3, \ldots$
- Exponential(λ) — mean $1/\lambda$; p.d.f. $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$; c.d.f. $F(x) = 1 - e^{-\lambda x}$
- $N(0, 1)$ — p.d.f. $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)$; c.d.f. $\Phi(z)$

In what follows we present in consecutive order six problems from MAT 135A, and six problems from MAT 135B.

PART I: Problems from Probability (MAT 135A)

1. A drawer contains 9 socks: 2 white, 2 black, 2 pink, 1 green, 1 yellow, 1 purple. Select 4 socks at random (without replacement).

   (a) Find the probability that only white and black socks are present.

   (b) Find the probability that 2 matching pairs are present.

   (c) Find the probability that at least one matching pair is present.

2. You have a small coin and a big coin. Each is biased, with a $1/3$ chance of Heads and $2/3$ chance of Tails. The small coin has "1" written on the Heads side and "0" written on the Tails side. The big coin has "3" written on the Heads side and "0" written on the Tails side. Every day you play a game: flip both coins, and win the sum of the two numbers shown (in dollars). For example, if both coins flip Tails then you win nothing, but if both coins flip Heads then you win \$4.

   (a) Find the expectation and variance of the amount you win on the first day.

   (b) You repeat this game every day for 100 days. Using the relevant approximation, estimate the probability that your total winnings exceed \$150.

   (c) Call a day "good" if you win at least \$1. Call a day "great" if you win \$4 (the maximum amount). Find the probability that the first good day is a great day.

3. Suppose 4 mathematicians and 4 physicists sit at random around a circular table.

   (a) Find the probability that each group sits together. (That is, the mathematicians sit in one contiguous block, and so do the physicists.)

   (b) Compute the expected number of mathematicians who have another mathematician sitting directly across the table. (Here, "directly across the table" means 4 seats to the left, or equivalently, 4 seats to the right.)

   (c) Compute the probability that every mathematician has another mathematician sitting directly across the table.

4. There are 4 doors, with 4 objects hidden behind them: a prize, two "stop" signs, and one "go" sign. There is exactly one object behind each door and they are assigned at random. You choose a door at random. If you find the prize, you win. If you find a "stop" sign, the game ends and you lose. If you find the "go" sign, you get to simultaneously open 2 of the remaining doors at random, and you win if you find the prize (otherwise you lose).

   (a) Find the probability of winning this game.

   (b) Conditional on winning the game, what is the probability of finding the "go" sign on the first move.

   (c) Say you win \$1 for each door opened, and if you find the prize you additionally win a bonus of \$5. Calculate your expected winnings.

5. Suppose X and Y have joint density

$$f(x,y) = \begin{cases} cy & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the value of c.

(b) Find the marginal density for X.

(c) Compute $\mathrm{E}[XY]$.

6. You are waiting at a bus stop. The arrival time of the next "A" bus is Exponential with expected value 10 minutes. The arrival time of the next "B" bus is Exponential with expected value 15 minutes, and independent from the A bus. You plan to take whichever bus (A or B) arrives first.

   (a) Find the probability that the A bus takes more than 10 minutes to arrive.

   (b) Find the probability that you wait at the bus stop for more than 10 minutes.

   (c) Find the probability that the B bus arrives more than 5 minutes after the A bus.

PART II: Problems from Stochastic Processes (MAT 135B)

1. Suspects $1, 2, \ldots$ are coming before a certain court. They are associated with i.i.d. random variables $X_1, X_2, \ldots$, continuously distributed with density

$$f(x) = \begin{cases} 2x & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Given $X_i = x$, suspect i is *guilty* with probability x (and *innocent* otherwise); one may imagine that guilt is determined by flip of a coin with Heads probability x. The court knows each X_i (the strength of evidence against i), but not whether any suspect is guilty (i.e., the coin flip is not observed). The court *convicts* some suspects based on the X_i and a fixed parameter $\alpha > 0$. (Note that "convicted" and "guilty" have different meanings.)

(a) Compute the long-term proportion of innocent people before the court.

(b) Assume the court convicts the suspect i exactly when $X_i > \alpha$. Compute the long-term proportion of innocent people among those that are convicted.

2. Initially (at 0th step), Alice has 3 cards and Bob none. Here are the rules at each step of the game. If one of them has no cards, then *redistribution* occurs: each card is given independently to Alice or Bob, with probability $\frac{1}{2}$. If both have at least one card, then one of the three cards is chosen at random; if the chosen card is in Alice's possession it is given to Bob, and if it is in Bob's possession it is given to Alice.

   (a) Determine the transition probability matrix of the Markov chain whose state is the number of cards Alice has.

   (b) Write down an expression for the probability that Alice has all the cards after 5 steps, but no cards after 10 steps.

   (c) Compute the invariant distribution for this chain (you may use symmetry to shorten your computations).

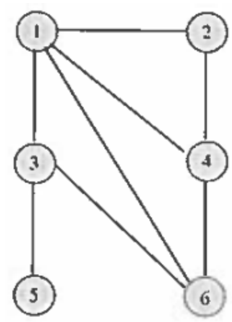
   (d) Compute the proportion of steps at which redistribution occurs.

3. Two random walkers walk on nonnegative integers. Alice starts at 0, while Bob starts at some large $n \geq 0$. Each second, each walker jumps to the right (i.e., adds a random number to the current position): Alice jumps by 2 or 3, each with probability 1/2; Bob jumps by 0 (probability $1 - p$), 1 (probability $p/2$), or 2 (probability $p/2$). Let p_n be the probability that the two ever meet (i.e., occupy the same point at some time).

   (a) Assume that $p = 0$, so that Bob never moves. Compute $\lim_{n\to\infty} p_n$.

   (b) Now assume that $p > 0$, and compute $\lim_{n\to\infty} p_n$. (*Hint*: difference between their positions.)

4. A random walker is at one of the six vertices, labeled $1, 2, 3, 4, 5, 6$, of the graph in the picture. At each time, she moves to a randomly chosen vertex connected to her current position by an edge. (All choices are equally likely and the walker never stays at the same position for two successive steps.)



(a) Compute the (long-run) proportion of time the walkerspends at each of the six vertices. Does this proportion depend on the walker's starting vertex?

(b) Compute the proportion of time the walker is at 1 and then at 2 in the next timestep.

(c) Assume that the walker starts at vertex 1. What is the expected time before the walker returns to 1?

5. In a branching process, an individual has no descendants (in the next generation) with probability $1 - p$, 1 descendant with probability $p/2$, and 3 descendants with probability $p/2$. The process starts with a single individual at generation 0. (Note that both answers below will depend on p.)

   (a) Compute the probability π_0 that the process ever goes extinct.

   (b) Compute the expected population size of the third generation.

6. A sensitive instrument is put online for a time period T (in hours). Two types of events, which occur as independent Poisson processes, may cause the instrument to malfunction: type A events occur at rate λ_A and type B events at rate λ_B.

   (a) Assume that (1) $T = 2$ and that (2) 2 or more events of either type in time interval $[0, T]$ will cause the malfunction. What is the probability of malfunction?

   (b) Assume that T is an Exponential random variable with $\mathrm{E}[T] = 2$, and keep the assumption (2) from (a). What is the probability of malfunction?

   (c) Assume (1) and (2) from (a). Given that exactly 1 type A event has occurred in time interval $[0, 1]$, what is the (conditional) probability of malfunction?

   (d) Assume that T is an Exponential random variable with $\mathrm{E}[T] = 2$, and the malfunction will happen if either at least 2 events of type A occur in $[0, T]$, or at least 1 event of type B occurs in $[0, T]$. What is the probability of malfunction?