

Applied Mathematics Preliminary Exam (September 15th, 2025)

First name : _____	Student ID : _____
Last name : _____	Additional pages : _____

Instructions:

1. Unless indicated, all problems are worth 10 points.
2. Explain your answers clearly. Unclear answers will not receive credit.
State results and theorems you are using.
3. Use the front and back of each page to write the solution of each problem.
4. If you need extra pages, please do not use the same sheet for different problems.
5. Write your name and the problem number in each additional page you use.

PROBLEM 1. Consider the nonlinear system of ODEs

$$\begin{aligned}\frac{dx}{dt} &= 2xy \\ \frac{dy}{dt} &= x^2 - y + \mu\end{aligned}$$

For $\mu = 0$,

- (a) Use linear stability analysis to assess the stability of the unique fixed point at $(0,0)$. What conclusions can be made about the stability of the fixed point?
- (b) Plot the nullclines of the system. Indicate the direction of the flow in the different regions of phase space. Sketch trajectories in the phase plane, and determine the stability of the fixed point at $(0,0)$ for the nonlinear system?

Now, for general μ ,

- (c) Plot the bifurcation diagram for the system, and classify the bifurcation that occurs at $\mu = 0$.
- (d) Determine whether or not any Hopf bifurcations occur in the system.

PROBLEM 2. Consider the nonlinear planar system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y)x - y \\ \frac{dy}{dt} &= x + g(x, y)y\end{aligned}$$

where $f(x, y)$ and $g(x, y)$ are smooth, bounded functions on $\mathbb{R}^2$. Furthermore, assume that

$$f(x, y) > 0 \text{ and } g(x, y) > 0 \text{ for all } (x, y) \text{ such that } x^2 + y^2 \leq a^2,$$

$$f(x, y) < 0 \text{ and } g(x, y) < 0 \text{ for all } (x, y) \text{ such that } x^2 + y^2 \geq b^2,$$

for some constants $0 < a < b$.

Show that, with one additional condition on $f(x, y)$ and $g(x, y)$, that the system is guaranteed to have a nontrivial stable periodic solution.

PROBLEM 3.

Consider the following boundary value problem of the second order ordinary differential equation:

$$\begin{cases} x^2 f'' + 3x f' + \lambda f = 0, & 1 \leq x \leq e \\ f(1) = 0 = f(e). \end{cases}$$

- (a) Using the substitution $x = e^t$ and $F(t) = f(x)$, convert this system into a regular Sturm-Liouville system in terms of $F(t)$.
- (b) Find the eigenvalues and eigenfunctions of the resulting RSL, and subsequently those of the original system. You do *not* need to normalize the eigenfunctions.

PROBLEM 4. Suppose $f \in C^1[-\pi, \pi]$, 2π -periodic, and real-valued.

- (a) Show that f' is orthogonal to f in $L^2(-\pi, \pi)$ using the fact that $(f^2)' = 2ff'$.
- (b) Show the same conclusion as Part (a) by expanding f in a Fourier series and using the following *Parseval's relation*:

$$\langle f, g \rangle = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle \overline{\langle g, e_n \rangle}, \quad \forall f, g \in L^2(-\pi, \pi),$$

where $\{e_n\}_{n \in \mathbb{Z}}$ is the Fourier orthonormal basis for $L^2(-\pi, \pi)$.

PROBLEM 5. Consider the boundary value problem:

$$\epsilon u'' + xu' - u = 0, \quad x \in [0, 1], \quad u(0) = a, \quad u(1) = b.$$

Use the matched asymptotic method to find the first term of the inner and outer expansions, and then derive a composite expansion of the solution of this problem for $\epsilon \ll 1$.

PROBLEM 6. Consider the initial value problem for the Duffing equation

$$\ddot{u} + u + \epsilon u^3 = 0, \quad t > 0, \quad u(0) = a, \quad \dot{u}(0) = b$$

Use two-timing method to find the leading asymptotic on the time scale $t = O(\epsilon^{-1})$ as $\epsilon \rightarrow 0$.