Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 20, 2011

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Let (X,d) be a metric space and let (x_n) be a sequence in X. For the purpose of this problem adopt the following definition: $x \in X$ is called a *cluster point* of (x_n) iff there exists a subsequence $(x_{n_k})_{k\geq 0}$ such that $\lim_k x_{n_k} = x$.

(a) Let $(a_n)_{n\geq 0}$ be a sequence of distinct points in X. Construct a sequence $(x_n)_{n\geq 0}$ in X such that for all $k = 0,1,2,\ldots$, a_k is a cluster point of (x_n).

(b) Can a sequence (x_n) in a metric space have an *uncountable* number of cluster points? Prove your answer. (If you answer yes, give an example with proof. If you answer no, prove that such a sequence cannot exists). You may use without proof that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.

Problem 2: (10 points)

Let X be a real Banach space and X^* its Banach space dual. For any bounded linear operator $T \in \mathscr{B}(X)$, and $\phi \in X^*$, define the functional $T^*\phi$ by

$$T^*\phi(x) = \phi(Tx), \quad \text{for all } x \in X.$$

(a) Prove that T^* is a bounded operator on X^* with $\|T^*\| \leq \|T\|$.

(b) Suppose $0 \neq \lambda \in \mathbb{R}$ is an eigenvalue of T. Prove that λ is also an eigenvalue of T^*. (**Hint 1:** first prove the result for $\lambda = 1$. **Hint 2:** For $\phi \in X^*$, consider the sequence of Cesàro means $\psi_N = N^{-1}\sum_{n=1}^{N}\phi_n$, of the sequence ϕ_n defined by $\phi_n(x) = \phi(T^n x)$.)

Problem 3: (10 points)

Let $\mathscr{H}$ be a complex Hilbert space and denote by $\mathscr{B}(\mathscr{H})$ the Banach space of all bounded linear transformations (operators) of $\mathscr{H}$ considered with the operator norm.

(a) What does it mean for $A \in \mathscr{B}(\mathscr{H})$ to be *compact*? Give a definition of compactness of an operator A in terms of properties of the image of bounded sets, e.g., the set $\{Ax \mid x \in \mathscr{H}, \|x\| \leq 1\}$.

(b) Suppose $\mathscr{H}$ is separable and let $\{e_n\}_{n\geq 0}$ be an orthonormal basis of $\mathscr{H}$. For $n \geq 0$, let P_n denote the orthogonal projection onto the subspace spanned by $e_0, \ldots, e_n$. Prove that $A \in \mathscr{B}(\mathscr{H})$ is compact iff the sequence $(P_n A)_{n\geq 0}$ converges to A in norm.

Problem 4: (10 points)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and smooth. Suppose that $\{f_j\}_{j=1}^{\infty} \subset L^2(\Omega)$ and $f_j \rightharpoonup g_1$ weakly in $L^2(\Omega)$ and that $f_j(x) \to g_2(x)$ a.e. in Ω. Show that $g_1 = g_2$ a.e. (**Hint:** Use Egoroff's theorem which states that given our assumptions, for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\lambda(E) < \epsilon$ and $f_j \to g_2$ uniformly on E^c.)

Problem 5: (10 points)

Let $u(x) = (1 + |\log x|)^{-1}$. Prove that $u \in W^{1,1}(0,1)$, $u(0) = 0$, but $\dfrac{u}{x} \not\in L^1(0,1)$.

Problem 6: (10 points)

Let $H = \left\{ f \in L^2(0,2\pi) \,:\, \int_0^{2\pi} f(x)\,\mathrm{d}x = 0 \right\}$. We define the operator Λ as follows:

$$(\Lambda f)(x) = \int_0^x f(y)\,\mathrm{d}y.$$

(a) Prove that $\Lambda : H \to L^2(0,2\pi)$ is continuous.

(b) Use the Fourier series to show that the following estimate holds:

$$\|\Lambda f\|_{H_0^1(0,2\pi)} \leq C\|f\|_{L^2(0,2\pi)},$$

where C denotes a constant which depends only on the domain $(0,2\pi)$. (Recall that $\|u\|^2_{H_0^1(0,2\pi)} = \int_0^{2\pi} \left|\dfrac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 \mathrm{d}x$.)

Problem 7: (10 points)

Consider the system

$$\dot{x} = \mu x + y + \tan x \qquad \dot{y} = x - y\ .$$

(a) Show that a bifurcation occurs at the origin $(x, y) = (0,0)$, and determine the critical value $\mu = \mu_c$ at which the bifurcation occurs.

(b) Determine the type of bifurcation that occurs at $\mu = \mu_c$. Do this (i) analytically and (ii) graphically (sketch the appropriate phase portraits for μ slightly less than; equal to; and slightly greater than μ_c).

Problem 8: (10 points)

Consider the differential equation

$$\ddot{x} + x - x^3 = 0\ ,$$

with the initial condition $x(0) = \epsilon$, $\dot{x}(0) = 0$, where $\epsilon \ll 1$. Use "two-timing" and perturbation theory to approximate the frequency of oscillation to order ϵ^2.

(a) Make a change of variables so that the differential equation is in the form $\ddot{z} + z + \epsilon h(z, \dot{z}) = 0$, i.e., in a form where ϵ appears naturally in the equation as a perturbation parameter.

(b) Rewrite the equation assuming two times scales, a fast time $\tau = t$ and a slow one $T = \epsilon\tau$, and the solution form $z(t,\epsilon) = z_0(\tau, T) + \epsilon z_1(\tau, T) + O(\epsilon^2)$.

(c) Show that the order 0 (i.e., $O(1)$) solution takes the form

$$z_0(\tau, T) = r(T)\cos(\tau + \phi(T))\ .$$

(d) Use the order 1 (i.e., $O(\epsilon)$) equation to determine the frequency of oscillation to order ϵ^2. (**Hint:** The order 1 (i.e., $O(\epsilon)$) equation contains resonant terms, which would cause the solution to grow without bound as $t \to \infty$. A solution that remains bounded for large τ is obtained by setting the coefficients of the resonant terms to zero. This yields equations that can be used to find the order ϵ^2 correction for the frequency of the oscillation. Note: Be sure to look for "hidden" resonance terms. It may be helpful to use the trig identity $\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)$.)