Probability Preliminary Exam
(June 18th, 2025)

First name : ______________	Student ID : ______________
Last name : ______________	Additional pages : ______________

Instructions:

1. **All problems are worth 10 points.**
2. **Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**
3. **Use the front and back of each page to write the solution of each problem.**
4. **If you need extra pages, please <u>do not</u> use the same sheet for different problems.**
5. **Write your name and problem number on each additional page you use.**
6. **Combinatorial expressions such as Binomial coefficients need not be simplified.**
7. **Answers should be in closed form, not containing an integral or summation sign.**
8. **Leave answers in terms of the standard Normal c.d.f. Φ when appropriate.**

Equations Provided:

- Poisson(λ) — mean λ; variance λ; p.m.f. $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$
- Geometric(p) — mean $1/p$; possible values $k = 1, 2, 3, \ldots$
- Exponential(λ) — mean $1/\lambda$; p.d.f. $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$; c.d.f. $F(x) = 1 - e^{-\lambda x}$
- $N(0,1)$ — p.d.f. $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)$; c.d.f. $\Phi(z)$

PROBLEM 1.

A pair (X, Y) of random variables has the joint density

$$f(x,y) = \begin{cases} cxy & \text{if } 0 \le x, y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c.

(b) Find the marginal densities of X and Y. Are X and Y independent?

(c) Find $\mathbf{P}(2Y < X \mid Y < X)$.

(d) Find $\mathbf{E}(X/Y)$.

PROBLEM 2.

A bag initially has 20 green balls and 10 red balls. Balls are repeatedly selected at random from the bag, without replacement, until the bag is empty.

(a) Find the conditional probability that the first 5 balls selected are green, given that the first 5 balls selected have the same color.

(b) Find the conditional probability that the first 5 balls selected are green, given that the last 5 balls selected are green.

PROBLEM 3.

In Sweden, 10 people are struck by lightning every year, on average. Answer the following questions about a given year in Sweden using a relevant approximation. (Assume person-to-person and day-to-day independence, and assume all months have the same length.)

(a) Find the probability that exactly one person is struck by lightning every month of the year.

(b) Find the probability that the first person struck by lightning is struck in the eighth month of the year.

(c) Say that a month is $good$ if no one is struck by lightning, and bad otherwise. Find the probability that the year has 5 good and 7 bad months.

(d) Find the probability that at some point in the year there are at least two bad months in a row.

PROBLEM 4.

5 Swedes, 5 Norwegians and 5 Finns are standing in a row in random order.

(a) Find the probability that all the Swedes are standing to the left of all the Finns.

(b) Say that a country is $happy$ if all of its citizens are standing together. Find the probability that Sweden is happy.

(c) Find the expected number of happy countries.

(d) Find the probability that there are no happy countries.

PROBLEM 5.

A branching process has a Binomial$(2, \frac{1}{2})$ offspring distribution. Start the process at generation 0 with a single individual and let X_n be the number of individuals in the nth generation.

(a) Find $\mathbf{E}(X_2)$.

(b) Find the probability that the branching process ever dies out.

(c) Find $\mathbf{E}(X_2 \mid X_2 > 0)$. Write your answer as a simple fraction.

PROBLEM 6.

Assume that cars arrive as a Poisson process of rate 6 per hour. Call a car $good$ if it will pick up a hitchhiker, and bad otherwise. Suppose that each car that arrives is good with probability $\frac{1}{2}$, independently of all other cars.

(a) Suppose you know that exactly 2 bad cars arrive arrive in the first hour. Compute the probability that exactly 2 good cars also arrived.

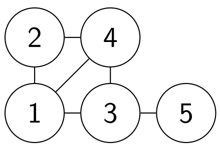
Two people, Alice and Bob, are hitchhiking. Alice is first in line for a ride.

(b) Find the expected length of time that Alice waits for a ride.

(c) Find the probability that Bob waits more than one hour for a ride.

PROBLEM 7.

Consider the random walk X_n on the following graph. At each step, the walker moves to each neighboring vertex with equal likelihood.



(a) Compute the long run proportion of time the walker spends at each of the five vertices. Does this proportion depend on the starting vertex?

(b) Suppose that states 1, 2, and 3 are *good* and states 4 and 5 are *bad*. Over the long run, what proportion of the time does the walker transition from a good state to a bad one?

(c) Suppose that $X_0 = 1$. Estimate the probability that X_{1000} and X_{2000} are both bad states.

PROBLEM 8.

Fix $p < \frac{1}{2}$ and consider the following random walk on the integers. If the current state is k, the next state is

$$\begin{cases} k+1 & \text{with probability } p; \\ k-1 & \text{with probability } 1-p. \end{cases}$$

Start the walk at 1. Compute the expected number of steps until the walk hits 0. (You may assume that the expectation is finite without any justification.)