Numerical Analysis Preliminary Exam
(September 16, 2025)

First name : ____________	Student ID : ____________
Last name : ____________	Additional pages : ____________

Instructions:

1. **All problems are worth 10 points.**
2. **Explain your answers clearly. Unclear answers will not receive credit.**
   State results and theorems you are using.
3. **Use the front and back of each page to write the solution of each problem.**
4. **If you need extra pages, please do not use the same sheet for different problems.**
5. **Write your name and problem number on each additional page you use.**

PROBLEM 1. Consider the data set

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\} = \{(-1, 0), (0, 0), (1/2, 3/4), (2, 6)\}.$$

(a) Construct the Lagrange form of the polynomial $p(x)$ that interpolates the given points.

(b) Compute the zeroth, first, second and third divided differences.

(c) Use your results in (b) to express $p(x)$ in Newton form.

(d) Suppose we wanted to add another interpolation point. Explain how you would update the Lagrange form of the interpolant and the Newton form of the interpolant. Which form requires fewer floating point operations to update?

PROBLEM 2. Recall the following theorem which allows you to bound the error between the integral of a function and its polynomial interpolant:

> *Theorem.* Let $g \in C^{(n+1)}([a,b])$ and let $x_0, x_1, \ldots, x_n$ be distinct points. Let p denote the unique polynomial interpolant to g of degree at most n at the points $x_0, x_1, \ldots, x_n$. Let $c, d \in \mathbb{R}$ such that $[c,d] \subset [a,b]$ and $\{x_0, x_1, \ldots, x_n\} \cap (c,d) = \emptyset$. Then, there exists a value $\xi \in (a,b)$ such that
>
> $$\int_c^d g(s)\,ds - \int_c^d p(s)\,ds = \frac{g^{(n+1)}(\xi)}{(n+1)!} \int_c^d w(s)\,ds,$$
>
> where
>
> $$w(x) = \prod_{k=0}^{n} (x - x_k).$$

(a) Use this result to show that if $f \in C^2([a,b])$ then

$$\int_a^b f(s)\,ds - T[f] = -\frac{(b-a)^3}{12} f''(\xi), \quad \xi \in [a,b],$$

where

$$T[f] = \frac{(b-a)}{2}(f(a) + f(b)).$$

(b) Based on (a), does $T[f]$ underestimate or overestimate $\int_a^b f(s)\,ds$ when f is convex?

(c) Show that $T[f]$ exactly integrates f when f is a degree 1 polynomial.

(d) Find coefficients $\alpha, \beta, \gamma \in \mathbb{R}$ such that the quadrature rule

$$\tau[f] = \alpha f(a) + \beta f\left(\frac{a+b}{2}\right) + \gamma f(b)$$

exactly integrates f when f is a degree 2 polynomial.

PROBLEM 3. Let $\theta \in [0, 2\pi)$. Define $G_\theta \in \mathbb{R}^{2\times 2}$ by

$$G_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

(a) Show that G_θ is orthogonal.

(b) Let $x \in \mathbb{R}^2$. Describe geometrically $G_\theta x$.

(c) Now, let n be a positive integer greater than 2. For any $\theta \in [0, 2\pi)$, $k \in \{1, 2, \ldots, n\}$, and $l \in \{1, 2, \ldots, n\}\backslash\{k\}$, define the matrix $G(k, l, \theta)$ by

$$G(k,l,\theta)_{ij} = \begin{cases} \cos(\theta) & i = j = k \text{ or } i = j = l \\ -\sin(\theta) & i = k,\ j = l \\ \sin(\theta) & i = l,\ j = k \\ 1 & i = j \neq k \text{ or } i = j \neq l \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, 2, \ldots, n$. A matrix of this form is called a *Givens rotation*. Show that $G(k, l, \theta)$ is orthogonal for any n.

(d) Briefly describe how you could compute a QR factorization of a general $n \times n$ matrix using Givens rotations.

PROBLEM 4. Let $A \in \mathbb{R}^{n\times n}$. Recall the definition of the *condition number* of A:

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2.$$

Suppose that $x, b \in \mathbb{R}^n$ satisfies

$$Ax = b$$

and $\widehat{x} \in \mathbb{R}^n$ satisfies

$$(A + \delta A)\widehat{x} = b,$$

where $\|\delta A\|_2 < \varepsilon \|A\|_2$ for some $\varepsilon > 0$. Show that if $\varepsilon\kappa_2(A) < 1$, then

$$\frac{\|x - \widehat{x}\|_2}{\|x\|_2} \leq \frac{\varepsilon\kappa_2(A)}{1 - \varepsilon\kappa_2(A)}.$$

Interpret this result.

PROBLEM 5. Let $y \in C^4(\mathbb{R})$.

(a) Use Taylor's theorem to show that for any $t \in \mathbb{R}$

$$y''(t) = \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} + O(h^2).$$

(b) Use Richardson extrapolation to find coefficients α_2, α_1, α_0, α_{-1}, and α_{-2} such that

$$y''(t) = \frac{\alpha_2 y(t+2h) + \alpha_1 y(t+h) + \alpha_0 y(t) + \alpha_{-1} y(t-h) + \alpha_{-2} y(t-2h)}{h^2} + O(h^4).$$

PROBLEM 6. In this question, you will derive the second-order Backward Differentiation Formula and analyze it. Recall that the BDF2 method applied to the differential equation

$$y'(t) = f(t, y(t)) \tag{1}$$

is given by

$$y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}hf(t_{n+1}, y_{n+1}), \quad n = 1, 2, \ldots. \tag{2}$$

(a) Derive (2). (*Hint: Let* p *be the polynomial that interpolates* y *at* t_{n-1}, t_n, t_{n+1}. *Consider* $p'(t_{n+1})$.)

(b) Compute the order of the truncation error for BDF2. In other words, find the largest q such that $T_{n+1}(y) = O(h^q)$ for $y \in C^\infty(\mathbb{R})$.

(c) Show that BDF2 is convergent.