207 Prelim Fall 2021

1. Consider the equation

$$\frac{d^2y}{dt^2} + \quad \frac{dy}{dt}\Bigg)^3 \quad 2\lambda\frac{dy}{dt} + y = 0$$

Convert this to a first order system of equations and investigate the Hopf bifurcation of this system near $\lambda = 0$. Determine the approximate bifurcation diagram in the $(\lambda, |y|)$ half-plane, near the origin.

2. Compare and contrast second order Hamiltonian and gradient systems. Let

$$F(x, y) = \quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{x^3}{3}$$

i) Consider a gradient system whose potential is given by $\vec{x} = (x, y)$, $V(\vec{x}) = F(x, y)$ and

$$\frac{d\vec{x}}{dt} = \quad \nabla V$$

Find the fixed points and analyze their linear stability. Plot contours of constant V. Explicitly find the fixed points, compute their linear stability and depict the eigenvectors on the plot of V. Sketch some trajectories on this plot.

ii) Show that there can be no oscillations in any gradient flow.

iii) Consider a Hamiltonian system whose Hamiltonian is given by $H(x, y) = F(x, y)$.

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = \quad \frac{\partial H}{\partial x}.$$

Find the fixed points and analyze their linear stability. Plot contours of constant H and sketch some trajectories on this plot.

iv) Find an approximation for the period of orbits near the stable fixed point.

v) Identify the homoclinic trajectory. What is the value of H on the homoclinic trajectory? How much time does it take to traverse the homoclinic orbit? Explain your answer.

3. i) Consider two variational principles

$$J_1 = \int^b L_1(x, u(x), u'(x))dx, \quad J_2 = \int^b L_2(x, u(x), u'(x))dx,$$

with $u(a) = u_1, u(b) = u_2$ fixed, where

$$L_2(x, u(x), u'(x)) = L_1(x, u(x), u'(x)) + \frac{d}{dx}F(x, u(x))$$

where F is any smooth function explicitly independent of u'. Prove that the two variational principles give rise to the same Euler-Lagrange equation.

ii) Suppose that the Euler-Lagrange equation is

$$\frac{d}{dx}(p(x)\frac{d}{dx}u(x)) + q(x)u(x) = 0.$$

Find a corresponding Lagrangian L.

iii) Let $a = 0, b = 1, p(x) = q(x) = 1$ in the Euler-Lagrange equation ii). Construct the Green's function.

4. Consider the di erential expression

$$Au = \frac{d^2}{dx^2}u + \frac{d}{dx}u + u$$

with the boundary condition $u(0) = u(1) = 0$.

i) Derive the inner product with respect to which A is self-adjoint and put A in the Sturm-Liouville form.

ii) Find all the eigenvalues $\{\lambda_n\}$ and their corresponding (normalized) eigenfunctions $\{\psi_n\}$.

iii) Find the solution u for $Au = 1$ in terms of $\{\psi_n\}$.

5. i) Find the leading asymptotic of the integral

$$I(x) = \int_x^\infty e^{\ \ t} \, dt, \quad x \to \infty.$$

(Hint: consider change of variables $s = t^2$.)

ii) Consider

$$u'' \quad u + \ u^3 = 0, \quad u(0) = 0, \ u(1) = 1$$

for $\ \ll 1$. Find the leading asymptotic u for u and write explicitly the equation, as well as the proper boundary conditions, that determine the next order asymptotic u_1 such that $u \sim u \ + u_1$.

6. Consider the boundary value problem

$$y'' + p(x)y' + q(x)y = 0, \quad x \in [a, b].$$

Find the location(s) of the boundary/internal layer(s) and the equation(s) governing the inner solution(s) for the following cases. (Your equations must be free of).

i) $p(x) = x, \quad q(x) = \ \ 1, \quad [a, b] = [\ \ 1, 1].$

ii) $p(x) = x^2, \quad q(x) = 1, \quad [a, b] = [0, 1].$