Fall 2019: Applied Mathematics Preliminary Exam

INSTRUCTIONS: All six questions are worth the same number of points. Explain your answers clearly, show all details (e.g., state all theorems you are using) and try to be organized. Unclear or unreadable answers will not receive any credit. Make sure to use separate sheets of paper for the solution of each problem.

1. Consider the planar system

$$\frac{dx}{dt} = x\,(1 - x - y)\,, \qquad \frac{dy}{dt} = y\left(3 - x - \frac{3}{2}y\right).$$

(a) Give an interpretation of this system in terms of the population dynamics of species $x(t) \geq 0$, $y(t) \geq 0$.

(b) Find the equilibria with $x, y \geq 0$, determine their linearized stability, and classify them.

(c) Sketch the phase plane of the system in $x \geq 0$, $y \geq 0$.

(d) Suppose that $x(0), y(0) > 0$. What happens to the solution as $t \to +\infty$? What is the interpretation in terms of population dynamics?

2. A discrete dynamical system for $x_n \in \mathbb{R}$ with $n = 0, 1, 2, \ldots$ depending on a parameter $\mu \in \mathbb{R}$ is given by

$$x_{n+1} = x_n^2 + \mu.$$

(a) Find the fixed points of the system and determine their linearized stability. Sketch a bifurcation diagram for the fixed points, indicating stable fixed points by a solid line and unstable fixed points by a dashed line.

(b) What types of bifurcations occur at the fixed points as μ increases from $-\infty$ to ∞?

3. Let

$$\mathcal{E}[y(x)] = \int_0^1 \left[\frac{dy}{dx}\right]^2 \, dx. \tag{1}$$

Find the functions, $y(x)$, for which $\mathcal{E}$ is extremal, subject to the constraints

$$\int_0^1 [y(x)]^2 \, dx = 1 \tag{2}$$

and

$$y(0) = 0, \quad y'(1) = 0 \tag{3}$$

where $y' = \frac{dy}{dx}$.

4. Let $\mathcal{D}$ be the disk of radius a in $\mathbf{R}^2$, i.e. $x^2 + y^2 \leq a^2$. Let $h(x, y)$ be a single valued function of x and y which describes a surface above $\mathcal{D}$ which is fixed on the boundary to

$$h(a\cos(\theta), a\sin(\theta)) = f(\theta).$$

(a) Find the functional $\mathcal{S}[h(x, y)]$ which computes the surface area of $h(x, y)$.

(b) Determine the equation for the function $h(x, y)$ which minimizes $\mathcal{S}[h(x, y)]$.

(c) Linearize the equation in (b) and solve it subject to the boundary condition

$$h(a\cos(\theta), a\sin(\theta)) = \cos^2(\theta).$$

5. Use the WKB method to find an approximate solution to the following problem

$$\begin{cases} \varepsilon y'' + 2y' + 2y = 0 \\ y(0) = 0 \\ y(1) = 1 \end{cases} \tag{4}$$

6. Consider the motion of a damped pendulum. The evolution of the position of the end of the pendulum $y(t)$ is described by,

$$\begin{cases} \varepsilon \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + \sin(y(t)) = 0 \\ y(0) = \varepsilon \\ y'(0) = 0 \end{cases} \tag{5}$$

where $\varepsilon \ll 1$. Use the method of multiple scales to find an approximate solution to the IVP above.

END OF EXAM.