Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
January 2, 2009

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Let $1 < p < 2$.

(a) Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.

(b) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, prove that $f \in L^p(\mathbb{R})$.

Problem 2: (10 points)

(a) State the Weierstrass approximation theorem.

(b) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 x^n f(x)\, dx = 0$$

for all non-negative integers n. Prove that $f = 0$.

Problem 3: (10 points)

(a) Define strong convergence, $x_n \to x$, and weak convergence, $x_n \rightharpoonup x$, of a sequence (x_n) in a Hilbert space $\mathscr{H}$.

(b) If $x_n \rightharpoonup x$ weakly in $\mathscr{H}$ and $\|x_n\| \to \|x\|$, prove that $x_n \to x$ strongly.

(c) Give an example of a Hilbert space $\mathscr{H}$ and sequence (x_n) in $\mathscr{H}$ such that $x_n \rightharpoonup x$ weakly and

$$\|x\| < \liminf_{n\to\infty} \|x_n\|.$$

Problem 4: (10 points)

Suppose that $T : \mathscr{H} \to \mathscr{H}$ is a bounded linear operator on a complex Hilbert space $\mathscr{H}$ such that

$$T^* = -T, \qquad T^2 = -I$$

and $T \neq \pm \mathrm{i}I$. Define

$$P = \frac{1}{2}(I + \mathrm{i}T), \qquad Q = \frac{1}{2}(I - \mathrm{i}T).$$

(a) Prove that P, Q are orthogonal projections on $\mathscr{H}$.

(b) Determine the spectrum of T, and classify it.

Problem 5: (10 points)

Let $\mathscr{S}(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R} \to \mathbb{C}$. Define an operator $H : \mathscr{S}(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\widehat{(Hf)}(\xi) = \mathrm{i}\,\mathrm{sgn}(\xi)\hat{f}(\xi) = \begin{cases} \mathrm{i}\hat{f}(\xi) & \text{if } \xi > 0, \\ -\mathrm{i}\hat{f}(\xi) & \text{if } \xi < 0, \end{cases}$$

where $\hat{f}$ denotes the Fourier transform of f.

(a) Why is $Hf \in L^2(\mathbb{R})$ for any $f \in \mathscr{S}(\mathbb{R})$?

(b) If $f \in \mathscr{S}(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = 0.$$

[Hint: you may want to use the Riemann-Lebesgue Lemma.]

Problem 6: (10 points)

Let Δ denote the Laplace operator in $\mathbb{R}^3$.

(a) Prove that

$$\lim_{\epsilon\to 0} \int_{B_\epsilon^c} \frac{1}{|\mathbf{x}|} \Delta f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 4\pi f(0), \quad \forall f \in \mathscr{S}(\mathbb{R}^3)$$

where B_ϵ^c is the complement of the ball of radius ϵ centered at the origin.

(b) Find the solution u of the Poisson problem

$$\Delta u = 4\pi f(\mathbf{x}), \qquad \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$$

for $f \in \mathscr{S}(\mathbb{R}^3)$.

Problem 7: (8 points)

Show that the solution to the system

$$\dot{x} = 1 + x^{10}$$

goes to infinity in finite time.

Problem 8: (12 points)

Consider the nonlinear system of ODEs:

$$\begin{aligned}\dot{x} &= y - x\left((x^2+y^2)^4 - \mu\left((x^2+y^2)^2 - 1\right) - 1\right)\\ \dot{y} &= -x - y\left((x^2+y^2)^4 - \mu\left((x^2+y^2)^2 - 1\right) - 1\right)\end{aligned}$$

(a) Rewrite the system in polar coordinates.

(b) For $0 \leq \mu < 1$, show that the circular region that lies within concentric circles with radius $r_{min} = 1/2$ and $r_{max} = 2$ is a trapping region. And use the Poincaré-Bendixson theorem to show that there exists a stable limit cycle.

(c) Show that a sub-critical Hopf Bifurcation occurs at $\mu = 1$.