

Applied Math

First name : _____	Student ID : _____
Last name : _____	Additional pages : _____

Instructions:

1. Explain your answers clearly. Unclear answers will not receive credit.
2. Use the front and back of each page to write the solution of each problem.
3. If you need extra pages, please do not use the same sheet for different problems.
4. Write your name and the problem number in each additional page you use.

Problem 1**(10 pts)**

Consider the one-dimensional dynamical system

$$\frac{dx}{dt} = f(x; \mu)$$

where f is smooth. Suppose that, for some (x_0, μ_0) ,

$$f(x_0; \mu_0) = 0, \quad f_x(x_0; \mu_0) = 0,$$

and

$$f_{xx}(x_0; \mu_0) = A \neq 0, \quad f_\mu(x_0; \mu_0) = B \neq 0.$$

- (a) Using a Taylor expansion of $f(x; \mu)$ about (x_0, μ_0) , identify the leading-order terms responsible for the bifurcation and show that the resulting system has the normal form of a saddle-node bifurcation.
- (b) Sketch the local bifurcation diagram. Indicate on which side of μ_0 the fixed points exist and the stability of each branch, in terms of the signs of A and B .
- (c) Explain the significance of the conditions $f_{xx}(x_0; \mu_0) = A \neq 0$ and $f_\mu(x_0; \mu_0) = B \neq 0$, and why each of these conditions is needed for a generic saddle-node bifurcation.

Problem 2**(10 pts)**

Consider the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \mu + x^2\end{aligned}$$

Note that this system can also be written as $\frac{d^2x}{dt^2} = \mu + x^2$.

- (a) Find all fixed points and determine their stability.
- (b) Sketch the bifurcation diagram and clearly describe the bifurcation that occurs. Explain why this is not a generic bifurcation for a general planar dynamical system.
- (c) Sketch all qualitatively distinct phase portraits of the system as μ is varied. Make sure to plot any separatrices, including the stable and unstable manifolds of any saddle-type fixed points.

Problem 3**(10 pts)**

Let $\alpha > 0$. Find the leading term in the asymptotic approximation of the integral

$$I(x) = \int_0^\alpha \exp \left[xt \left(1 - \frac{t^2}{\alpha^2} \right) \right] dt$$

in each of the following two limits:

(a) $x \rightarrow -\infty$;

(b) $x \rightarrow \infty$.

Problem 4**(10 pts)**

Let $a > 0$ be fixed. Use boundary-layer asymptotic analysis to find the leading-order uniformly valid composite approximation to the solution of the boundary-value problem

$$-\varepsilon \frac{d^2 y}{dx^2} + (a^2 + x^2) \frac{dy}{dx} - xy = 0, \quad 0 < x < a,$$

subject to the boundary conditions

$$y(0) = 1, \quad y(a) = 0,$$

where

$$0 < \varepsilon \ll 1.$$

Sketch the resulting approximation $y(x)$, clearly indicating the location of the boundary layer and stating its thickness as a power of ε .

Problem 5**(10 pts)**

On the complex Hilbert space $L^2(0, 1)$, consider the integral operator

$$(Ku)(t) = \int_0^1 k(t, s)u(s) ds, \quad k(t, s) = 2 + 4 \cos(2\pi(t - s)) - 2 \cos(4\pi(t - s)).$$

- (a) Find all nonzero eigenvalues of K and their eigenspaces. State the multiplicity of each nonzero eigenvalue, and describe both $\ker K$ and $\text{Ran } K$.
- (b) For $a \in \mathbb{R}$, find the unique minimum-norm least-squares solution of

$$Ku(t) = 1 + \sin(2\pi t) + \cos(4\pi t) + a \cos(6\pi t).$$

Then describe all least-squares solutions.

- (c) For which value or values of a is a least-squares solution in part (b) an exact solution?

Problem 6**(10 pts)**For $\lambda \in \mathbb{R}$, let

$$L_\lambda u = u'' + \lambda u, \quad 0 < x < 1,$$

with the mixed homogeneous boundary conditions

$$u'(0) = 0, \quad u(1) = 0.$$

(a) Construct the Green function $G_\lambda(x, t)$ satisfying

$$(\partial_x^2 + \lambda)G_\lambda(x, t) = \delta(x - t), \quad \partial_x G_\lambda(0, t) = 0, \quad G_\lambda(1, t) = 0,$$

for every value of λ for which it exists. Your answer should cover the three cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.**(b)** Determine all real values of λ for which the Green function does not exist.**(c)** Fix one of the values $\lambda = \lambda_n$ found in part (b). Give a necessary and sufficient condition on $f \in L^2(0, 1)$ for the boundary-value problem

$$L_{\lambda_n} u = f, \quad u'(0) = 0, \quad u(1) = 0,$$

to have a solution. When the condition holds, explain why the solution is not unique and describe the full family of solutions.