GGAM PRELIMINARY EXAM
**January 3, 2005**

**Write solutions on the paper provided, putting each problem on a separate page. Justify all of your mathematics. Print your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.**

**1.** Consider the equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \tag{1}$$

with parameter $\mu$.

(a) Show that equation (1) is equivalent to the system

$$\begin{aligned} \dot{x}(t) &= y - g(x) \\ \dot{y}(t) &= -x, \end{aligned} \tag{2}$$

with $g(x) = \mu(\frac{1}{3}x^3 - x)$.

(b) Consider the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of system (2). Calculate the time derivative of $V(x(t), y(t))$. Describe the regions where $\frac{dV}{dt}$ is negative and positive. (These regions should depend on $\mu$.)

(c) For which values of $\mu$ does the fixed point $(0, 0)$ change its stability?

(d) For which values of $\mu$ does the system have a limit cycle? Explain your answer using the Poincare-Bendixson Theorem.

**2.** Consider the system

$$\dot{x} = x[x(1 - x) - y] \qquad \dot{y} = y(x - 2), \tag{3}$$

where $x \geq 0$ and $y \in \mathbb{R}$.

(a) What is the asymptotic behavior, as $t \to \infty$, of the trajectory starting at $(2, 1)$?

(b) What is the asymptotic behavior, as $t \to \infty$, of the trajectory starting at $(2, -4)$? (Explain your answers.)

**3.** Suppose that $(P_n)$ is a sequence of orthogonal projections on a Hilbert space $\mathcal{H}$ such that

$$\operatorname{ran} P_{n+1} \supset \operatorname{ran} P_n, \qquad \bigcup_{n=1}^{\infty} \operatorname{ran} P_n = \mathcal{H},$$

where $\operatorname{ran} P_n$ denotes the range of $P_n$.

(a) Prove that for every $x \in \mathcal{H}$, the sequence $(P_n x)$ converges to $x$ as $n \to \infty$ with respect to the norm on $\mathcal{H}$.

(b) Prove that $(P_n)$ does not converge to the identity operator $I$ with respect to the operator norm on $\mathcal{B}(\mathcal{H})$ unless $P_n = I$ for some $n$.

**4.** If $A$ is a subset of a metric space $X$, with metric $d : X \times X \to \mathbb{R}$, define the 'distance from $A$' function $f_A : X \to \mathbb{R}$ by

$$f_A(x) = \inf_{a \in A} d(x, a).$$

(a) Prove that $f_A$ is continuous.

(b) Prove that if $A$ is a closed subset of $X$, then

$$A = \{x \in X \mid f_A(x) = 0\}.$$

(c) A subset of a metric space is said to be a $G_\delta$ if it is a countable intersection of open sets. Prove that every closed subset of a metric space is a $G_\delta$.

**5.** Suppose that $f : [0,1] \to \mathbb{R}$ is a function with the property that there exist constants $M > 0$, $\alpha > 1$ such that

$$|f(x) - f(y)| \le M|x - y|^\alpha \qquad \text{for all } x, y \in [0,1],$$

Prove that $f$ is a constant.

**6.** (a) Briefly define a distribution on $\mathbb{R}$, and define what it means for a sequence $(T_n)$ of distributions to converge.

(b) Show that the following sequence converges in the sense of distributions, and determine its limit:

$$T_n = \sin(nx).$$

(c) Show that the following sequence converges in the sense of distributions, and determine its limit:

$$T_n = n\left(\delta_{1/n} - \delta_{-1/n}\right).$$

Here, $\delta_a$ denotes the delta-distribution supported at $a$, defined by

$$\langle \delta_a, \phi \rangle = \phi(a).$$

**7.** (a) Define the subspace $H^1([0,1])$ of $L^2[0,1]$ using Fourier series.

(b) Give the definition of the *spectrum* of a linear operator $A$ (not necessarily bounded) defined on a (dense linear) domain $\mathcal{D}(A)$ in a Hilbert space $\mathcal{H}$.

(c) Let $A$ be the (unbounded) linear operator on $\mathcal{H} = L^2([0,1])$ with domain $\mathcal{D}(A) = \{u \in H^1([0,1]) \mid u(0) = u(1)\}$, and $Au = \frac{1}{2\pi i} u'$, where $u'$ is the weak derivative of $u$. Prove that the spectrum of $A$ defined above is the set of integers.