Applied Math Prelim Exam (Fall 2014)

Instructions.

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State all results and theorems that you are using.
2. Use separate sheets for the solution of each problem.

1. Consider the mechanical system pictured below. A particle attached to a spring, of rest length 1, slides along a rigid rod. The rigid rod is situated a distance h and at an angle θ from the surface to which the spring is attached.

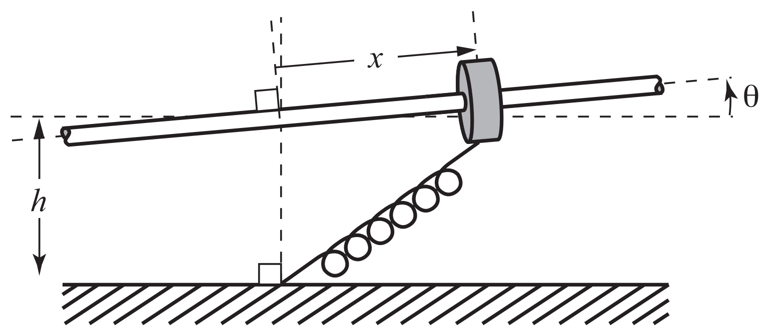

FIGURE 1. A mechanical system.

Assuming that damping is large, the differential equation that governs the particle's position is of the form

$$\frac{dx}{dt} = \left(\frac{1}{\sqrt{h^2 + 2xh\sin(\theta) + x^2}} - 1 \right) (x + h\sin(\theta)) .$$

There are two parameters in the equation, θ and h. Sketch a bifurcation diagram in h for the case where $\theta = 0$. Then, sketch a bifurcation diagram in θ for the case where $h = 1 + \varepsilon$ (where ε is an arbitrarily small, but non-zero, positive number). Finally, sketch a stability diagram in h and θ, and find an equation for the boundaries between the phases. In all of your answers, assume that $h \geq 0$ and $-\pi/2 < \theta < \pi/2$.

2. A generic conservative, one degree-of-freedom mechanical system obeys the following differential equations

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -g(x_1)$$

Suppose that there is a local, isolated minimum of the potential energy function $V(x_1) = \int_0^{x_1} g(x_1)$ at x_1^*. Show that this minimum at x_1^* corresponds to a stable equilibrium at $x_1 = x_1^*$, $x_2 = 0$. You may assume that g is C^1.

Note. Recall that a fixed point $\bar{x}$ is asymptotically stable if all nearby trajectories converge to $\bar{x}$ as time $t \to \infty$; it is Lyapanov stable if nearby trajectories remain close to $\bar{x}$ for all time. In this problem, by "stability," we refer to either type.

3. Find the path between (x_1, y_1) and (x_2, y_2) which a particle sliding without friction and under constant gravitational acceleration will traverse in the shortest time. You may assume that the particle is released from (x_1, y_1) at rest and hence conservation of energy implies that

$$\frac{1}{2}mv^2 + mgy = mgy_1.$$

4. Determine the Green's function associated with the BVP

$$x^2y'' - xy' - 3y = x - 3, \quad y(1) = 0, y(2) = 0,$$

and give a solution to the BVP.

5. Find the leading order approximations in the limits $x \to \infty$ and $x \to -\infty$ of

$$\int_0^\pi e^{x\sin(t)}\,dt\,.$$

6. For $0 < \epsilon \ll 1$ and $k(\epsilon x) > 0$, $\forall x \in \mathbb{R}$, with $k(\epsilon x) \sim \mathcal{O}(1)$, consider the following two second order ODEs:

$$\frac{d}{dx}\left[\frac{1}{k(\epsilon x)^2}\frac{dh_1}{dx}\right] + h_1 = 0 \tag{1}$$

$$\frac{1}{k(\epsilon x)^2}\frac{d^2h_2}{dx^2} + h_2 = 0\,. \tag{2}$$

Equation (1) describes the amplitude $h_1(x)$ of a wave in a medium with varying wave speed, while equation (2) describes the amplitude $h_2(x)$ of a harmonic oscillator with varying frequency.

Compute the WKB approximation (up to $\mathcal{O}(\epsilon)$, i.e., two terms in the asymptotic expansion) for both equations (1) and (2). Furthermore, assuming that

$$\lim_{x\to-\infty} k \to k_- \quad \text{and} \quad \lim_{x\to\infty} k \to k_+\,,$$

and that

$$\lim_{x\to-\infty} |h_1(x)|^2 = 1 \text{ and } \lim_{x\to-\infty} |h_2(x)|^2 = 1,$$

determine the limits

$$\lim_{x\to\infty} |h_1(x)|^2 \text{ and } \lim_{x\to\infty} |h_2(x)|^2\,.$$