

Probability Preliminary Exam (September 2026)

First name : _____	Student ID : _____
Last name : _____	Additional pages : _____

Instructions:

1. All problems are worth 10 points.
2. Explain your answers clearly. Unclear answers will not receive credit.
State results and theorems you are using.
3. Use separate sheets for the solution of each problem.
4. Combinatorial expressions such as Binomial coefficients need not be simplified.
5. Answers should be in closed form, not containing an integral or summation sign.
6. Leave answers in terms of the standard Normal c.d.f. Φ when appropriate.

Equation Sheet

- Poisson(λ) — mean λ ; variance λ ; p.m.f. $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \dots$
- Geometric(p) — mean $1/p$; possible values $k = 1, 2, 3, \dots$
- Exponential(λ) — mean $1/\lambda$; p.d.f. $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$; c.d.f. $F(x) = 1 - e^{-\lambda x}$
- $N(0, 1)$ — p.d.f. $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$; c.d.f. $\Phi(z)$
- Markov's inequality: if $X \geq 0$ and $a > 0$ then $P(X \geq a) \leq \mathbf{E}[X]/a$
- Chebyshev's inequality: for $k > 0$, $P(|X - \mathbf{E}[X]| \geq k) \leq \text{Var}(X)/k^2$
- Deck of cards: 4 suits (spades, hearts, diamonds, clubs), 13 values (2–10, jack, queen, king, ace); total number of cards is $4 \times 13 = 52$ (no jokers)

PROBLEM 1.

Shuffle a standard deck of 52 cards and then reveal all the cards one by one. Say a card “follows a spade” if the card directly before it is a spade.

- (a) Find the probability that the king of hearts follows a spade.
- (b) Find the probability that the king of hearts follows a spade, or the queen of hearts follows a spade (or both).
- (c) Find the expected number of hearts that follow a spade.
- (d) Given that the queen of hearts follows a spade, find the conditional probability that the king of hearts follows a spade.

PROBLEM 2.

You are standing on a street corner, watching traffic pass. On average, a bike passes every 2 minutes. On average, a car passes every 4 minutes. Assume the arrival time of the next bike is Exponential, and similarly for cars. Also assume that bikes and cars are independent from each other.

- (a) Find the probability that no vehicles pass during the first 6 minutes. (Both bikes and cars count as “vehicles.”)

- (b) Find the probability that no cars pass during the first 5 minutes but then at least one car passes during the next 5 minutes.

- (c) Find the probability that the first vehicle to pass is a bike.

- (d) Find the probability that two (or more) cars pass during the first 10 minutes.
For this part, it is ok to leave integral sign(s) in your answer.

PROBLEM 4.

A Markov chain on $\{1, 2, \dots, 10\}$ has the following transition rule. If the current state is k , the next state is

$$\begin{cases} k + 1 & \text{if } k < 10; \\ \text{chosen uniformly from } \{1, \dots, 10\} & \text{if } k = 10. \end{cases}$$

Find the stationary distribution π of this chain. (Hint: it is possible to do this without long calculations. You might find the following fact useful: $1 + 2 + \dots + 10 = 55$.)

PROBLEM 5.

You roll a fair die repeatedly.

- (a) Let T be the number of rolls until the pattern 1234 first appears. Find $\mathbf{E}(T)$. (Hint: consider the appropriate Markov chain.)

- (b) Let W be the number of rolls until the pattern 6666 first appears. Find $\mathbf{E}(W)$. (Hint: it's not the same answer as part (a).)

PROBLEM 6.

Fix $p < \frac{1}{2}$ and consider the following random walk on the integers. If the current state is k , the next state is

$$\begin{cases} k + 1 & \text{with probability } p; \\ k - 1 & \text{with probability } 1 - p. \end{cases}$$

Start the walk at 1. Compute the expected number of steps until the walk hits 0. (You may assume that the expectation is finite without any justification.)