Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
April 1, 2010

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Let (X, d) be a complete metric space, $\bar{x} \in X$, and $r > 0$. Set $D := \{x \in X : d(x, \bar{x}) \le r\}$, and let $f : D \to X$ satisfying

$$d\left(f(x), f(y)\right) \le k\, d(x, y)$$

for any $x, y \in D$, where $k \in (0, 1)$ is a constant.

Prove that if $d\left(\bar{x}, f(\bar{x})\right) \le r(1-k)$, then f admits a unique fixed point. (Guidelines: Assume the Banach fixed point theorem, also known as the contraction mapping theorem.)

Problem 2: (10 points)

Give an example of two normed vector spaces, X and Y, and of a sequence of operators, $\{T_n\}_{n=0}^{\infty}$, $T_n \in L(X, Y)$ (where $L(X, Y)$ is the space of the continuous operators from X to Y, with the topology induced by the operator norm) such that $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence but it does not converge in $L(X, Y)$. (Notice that Y cannot be a Banach space otherwise $L(X, Y)$ is complete.)

Problem 3: (10 points)

Let (a_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^3$$

converges. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

also converges.

Problem 4: (10 points)

Suppose that $h : [0,1]^2 \to [0,1]^2$ is a continuously differentiable function from the square to the square with a continuously differentiable inverse h^{-1}. Define an operator T on the Hilbert space $L^2([0,1]^2)$ by the formula $T(f) = f \circ h$. Prove that T is a well-defined bounded operator on this Hilbert space.

Problem 5: (10 points)

Let $H^s(\mathbb{R})$ denote the Sobolev space of order s on the real line $\mathbb{R}$, and let

$$\|u\|_s := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

denote the norm on $H^s(\mathbb{R})$, where $\hat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ix\xi} \, dx$ denotes the Fourier transform of u.

Suppose that $r < s < t$, all real, and $\epsilon > 0$ is given. Show that there exists a constant $C > 0$ such that

$$\|u\|_s \leq \epsilon \|u\|_t + C\|u\|_r \quad \forall u \in H^t(\mathbb{R}).$$

Problem 6: (10 points)

Let $f : [0,1] \to \mathbb{R}$. Show that f is continuous if and only if the graph of f is compact in $\mathbb{R}^2$.

Problem 7: (10 points)

The precession of the perihelion of a planet in Einstein's Theory of General Relativity. In our solar system, General Relativity can be viewed as a small perturbation to the regular Newtonian theory of gravity. When studying the problem of planets going around the sun, you get an equation for $u = r^{-1}$ where r is the distance from the planet to the center of mass of the system and θ is the angle of the planet in its orbit. In General Relativity, the equivalent equation is approximately

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{1}{L} + \epsilon L u^2$$

where L is related to the angular momentum and $0 < \epsilon \ll 1$. When $\epsilon = 0$, this is the equation for Newtonian gravity.

(a) What does a circular orbit correspond to in this system?

(b) Find the fixed points and classify their stability. (There is a center and a saddle). Also, expand the location of the center in a Taylor series in ϵ, retaining only the first two terms.

(c) Sketch the phase portrait and identify the region where there are periodic solutions in θ.

(d) When $\epsilon = 0$, what is the period of all of the closed orbits, $r(\theta)$? What does this mean for the shape of the orbits in physical space (i.e., what do the ACTUAL planetary trajectories look like and how is this related to the result you just found?). Sketch one orbit.

(e) Find and approximate expression for the period of nearly circular orbits when $0 < \epsilon \ll 1$. What does this mean for the shape of the orbits in physical space? Sketch two periods of such an orbit.

Problem 8: (10 points)

Consider the system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x(a - x - y)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2a(y - x) + y(x - y).\end{aligned}$$

(a) Find the equilibrium points and for what values of a they exist.

(b) Find the linear stability of each point as a function of a.

(c) Sketch the phase portrait for representative values of a.

(d) Sketch the bifurcation diagram in the (a, x)-plane.