Spring 2022, 207 Applied Mathematics Preliminary Exam

INSTRUCTIONS: All six questions are worth the same number of points. Explain your answers clearly, show all details (e.g., state all theorems you are using) and try to be very organized and neat. Unclear or unreadable answers will not receive any credit. Make sure to use separate sheets of paper for the solution of each problem. Put your name in each sheet of paper.

When you are finished, please upload your work into GRADESCOPE. Make sure to upload separate answers for each of the six questions. Return your papers, including scratch paper, to proctor. Make sure to present in the right order of pages and to include any scratch paper at the very end with a label.

1. Use Green's theorem

$$\oint_C \mathbf{v}\cdot\mathbf{n}\, d\, l = \int\int_A \nabla\cdot\mathbf{v}\, dA$$

to understand the structure of the following dynamical system.

(a) The van der Pol oscillator is

$$\ddot{x} + \epsilon\, \dot{x}\left(x^2 - 1\right) + x = 0.$$

where we will assume a weak damping term, $0 < \epsilon << 1$. Assume that the limit cycle is a circle of unknown radius, a, about the origin. By writing the vector field associated with the van der Pol oscillator and using Green's theorem, determine the radius of the limit cycle. ($\mathbf{v} = (\dot{x}, \dot{y})$).

(b) Show that if the divergence of the vector field does not vanish in a domain, D, then there can be no closed orbit in D.

(c) Does the dynamical system

$$\begin{aligned} \dot{x} &= 2x - xy^2 + \cos(y) \\ \dot{y} &= -y - x^2 y + \sin(x) \end{aligned}$$

have any periodic orbits within the circle of radius one, centered at the origin?

2. Consider

$$\begin{aligned} \frac{dx}{dt} &= a(3x - 5y) - x^2 + y^2 \\ \frac{dy}{dt} &= x(2a - y) \end{aligned}$$

(a) Find the equilibrium points and for what values of a they exist.

(b) Find the linear stability of each point as a function of a.

(c) Sketch the phase portrait for representative values of a.

(d) Sketch the bifurcation diagram in the (a, x)-plane.

3. For the differential expression

$$Au = \frac{d^2}{dx^2}u - \frac{d}{dx}u, \quad x \in (0, 1),$$

find the Green function **separately** with the following boundary conditions.

i) (5 pt) $u(0) = u(1) = 0$;

ii) (5 pt) $u'(0) = u'(1) = 0$.

4. Consider the Laplacian Δ in $D = [0, a] \times [0, b]$ with the boundary conditions

$$u(x, 0) = u(x, b) = 0; \quad u_x(0, y) = u_x(a, y) = 0, \quad \forall x \in [0, a],\ y \in [0, b].$$

i) (5 pt) Find the complete sets of eigenvalues and eigenfunctions.

ii) (5 pt) Solve the equation

$$\Delta u(x, y) = x, \quad (x, y) \in D$$

in terms of the eigenfunction expansion.

5. (10 pt) Find the leading asymptotic of the following integrals

$$\int_0^1 e^{-xt} \sin t \ dt; \qquad \int_0^1 e^{-xt} t^{-1/2} dt$$

as $x \to \infty$.

6. (10 pt) Consider the equation

$$\epsilon u'' + u' = c, \quad x \in (0, 1),$$

where $u(0) = 0, u(1) = 1$, and $c \neq 1$ is a given positive constant. After finding the first term of the inner and outer expansions for $\epsilon \to 0$, derive a composite expansion of the solution of this problem.