Fall 2006: PhD Applied Math Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a given continuous function. Let $T_k : C([0,1]) \to C([0,1])$ be the linear operator given by $T_k(f)(x) = \int_0^1 k(x,y) f(y)\, dy$.

(a) Show that T_k is a bounded operator.

(b) Find an expression for $\|T_k\|$ in terms of k.

(c) What is $\|T_k\|$ if $k(x,y) = x^2 y^3$?

Problem 2. Let X be a metric space.

(a) Define X is sequentially compact.

(b) Define X is a complete metric space.

(c) Prove that a sequentially compact metric space X is complete.

(d) Let $B = \{ x : \|x\|_2 \leq 1 \}$ be the unit ball in $\ell^2(\mathbb{N})$. Show that B is not sequentially compact.

Problem 3. Give an example of a Banach space X and a sequence (x_n) of elements in X such that $\sum_{n=1}^\infty x_n$ converges unconditionally (converges regardless of order), but does not converge absolutely ($\sum_{n=1}^\infty |x_n|$ does not converge). Prove this.

Problem 4. Let $f \in L^2(\mathbb{T})$, and let $(\hat{f}_n)_{n \in \mathbb{Z}}$ be the Fourier coefficient sequence of f; here, $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$. If $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, does it follow that f is continuous? (In other words, is there a continuous function that is equivalent to f in $L^2(\mathbb{T})$?) Prove your assertion.

Problem 5. Find all solutions T of the equation $x^{2006}T = 0$ in the space of tempered distributions $S^*(\mathbb{R}^1)$.

Problem 6. In which of the following cases is the operator $A = i\frac{d}{dx}$ acting on $L^2([0,1])$ *symmetric*, *essentially self-adjoint*, *self-adjoint*? Justify your answers.

(a) $D_A = C^1[0,1]$
(the space of continuously differentiable complex-valued functions on $[0,1]$)

(b) $D_A = \{\, f \in C^1[0,1] \ : \ f(0) = f(1)\,\}$

(c) $D_A = \{\, f \in C^1[0,1] \ : \ f(0) = f(1) = 0\,\}$

Problem 7. Consider the system

$$\begin{aligned} \dot{x} &= -y + axe^{x^2+y^2} \\ \dot{y} &= x + aye^{x^2+y^2} \end{aligned}$$

near the fixed point $(0,0)$, where a is a parameter.

(a) Classify the stability of the fixed point $(0,0)$ in its linearized system.

(b) Classify the stability of the origin in the original nonlinear system. (Hint: Express the system in polar coordinates, and recall that $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$.)

Problem 8. Show that the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y\left(4 - x^2 - 4y^2\right) \end{aligned}$$

has at least one closed orbit in the annulus

$$1 \le x^2 + y^2 \le 4.$$