Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 21, 2010

Instructions:

- *This exam has 4 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Let $f(x,y)$ denote a C^1 function on $\mathbb{R}^2$. Suppose that

$$f(0,0) = 0.$$

Prove that there exist two functions, $A(x,y)$ and $B(x,y)$, both continuous on $\mathbb{R}^2$ such that

$$f(x,y) = xA(x,y) + yB(x,y) \quad \forall (x,y) \in \mathbb{R}^2$$

(**Hint**: Consider the function $g(t) = f(tx,ty)$ and express $f(x,y)$ in terms of g via the fundamental theorem of calculus .)

Problem 2: (10 points)

The Fourier transform $\mathscr{F}$ of a distribution is defined via the duality relation

$$\langle \mathscr{F} f, \phi \rangle = \langle f, \mathscr{F}^* \phi \rangle$$

for all $\phi \in C_0^\infty(\mathbb{R})$, the smooth compactly-supported test functions on $\mathbb{R}$, where

$$\mathscr{F}^* \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \phi(\xi)\, d\xi .$$

Explicitly compute $\mathscr{F} f$ for the function

$$f(x) = \begin{cases} x, & x > 0 \\ 0, & x \le 0 \end{cases} .$$

Problem 3: (10 points)

Let $\{P_n(x)\}_{n=1}^{\infty}$ denote a sequence of polynomials on $\mathbb{R}$ such that

$$P_n \to 0 \text{ uniformly on } \mathbb{R} \text{ as } n \to \infty.$$

Prove that, for n sufficiently large, all P_n are constant polynomials.

Problem 4: (10 points)

For $g \in L^1(\mathbb{R}^3)$, the convolution operator G is defined on $L^2(\mathbb{R}^3)$ by

$$Gf(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} g(x-y) f(y)\,\mathrm{d}y, \quad f \in L^2(\mathbb{R}^3).$$

Prove that the operator G with

$$g(x) = \frac{1}{4\pi} \frac{\mathrm{e}^{-|x|}}{|x|}, \quad x \in \mathbb{R}^3,$$

is a bounded operator on $L^2(\mathbb{R}^3)$, and the operator norm $\|G\|_{op} \le 1$.

Problem 5: (10 points)

Consider the map which associates to each sequence $\{x_n : n \in \mathbb{N}, x_n \in \mathbb{R}\}$ the sequence, $\{(F(\{x_n\}))_m ; m \in \mathbb{N}, (F(\{x_n\}))_m \in \mathbb{R}\}$, defined as follows:

$$\Big\{F(\{x_n\})\Big\}_m := \frac{x_m}{m} \quad \text{for} \quad m = 1, 2, \ldots$$

1. Determine (with proof) the values of $p \in [1, \infty]$ for which the map $F : l^p \to l^1$ is well-defined and continuous.
2. Next, determine the values of $q \in [1, \infty]$ for which the map $F : l^q \to l^2$ is well-defined and continuous.

Note for $1 \leq p < \infty$, l^p denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, while l^∞ denotes the space of sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sup_{n \in \mathbb{N}} |x_n| < \infty$.

Problem 6: (10 points)

For each of the following, determine if the statement is true (always) or false (not always true). If true, give a brief proof, e.g., by citing a relevant theorem; if false, give a counterexample.

Let $\mathbb{H}$ denote a separable Hilbert space and (x_n) a sequence of $\mathbb{H}$.

(a) If (x_n) is weakly convergent then it is strongly convergent.

(b) If (x_n) is strongly convergent then it is bounded.

(c) If (x_n) is weakly convergent then it is bounded.

(d) If (x_n) is bounded, there exists a strongly convergent subsequence of (x_n).

(e) If (x_n) is bounded, there exists a weakly convergent subsequence of (x_n).

(f) If (x_n) is weakly convergent and T is a bounded linear operator from $\mathbb{H}$ to $\mathbb{R}^d$, for some d, then $T(x_n)$ converges in $\mathbb{R}^d$.

Problem 7: (10 points)

Consider the first order ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta + \alpha x - x^3 = f(x)\,,$$

with $\alpha, \beta \in \mathbb{R}$.

(a) What conditions on $f(x)$ and $f'(x)$ must be satisfied simultaneously at bifurcation points? *Briefly* explain your answers.

(b) Use these conditions to find the curves of bifurcation points in α vs. β parameter space? Sketch the corresponding curves in the α vs. β plane.

(c) Sketch the following bifurcation diagrams. Indicate the stability of the fixed points on the diagrams and classify the bifurcations that occurs. A detailed analytical treatment of the system is not required, but some justification (e.g., graphical arguments) of your answers is required.

(i) Use α as the bifurcation parameter and hold β constant at $\beta = 0$.

(ii) Use α as the bifurcation parameter and hold β constant with $\beta > 0$.

(iii) Use β as the bifurcation parameter and hold α constant with $\alpha > 0$.

Problem 8: (10 points)

Consider the system of ordinary differential equations in polar coordinates

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(1 - r^2)(4 - r^2), \qquad r \geq 0$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2 - r^2.$$

Sketch the phase-portrait. Label all fixed points and limit cycles, and indicate their stability.