Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 23, 2008

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Prove that the dual space of c_o is ℓ^1, where

$$c_o = \{x = (x_n) \text{ such that } \lim x_n = 0\}.$$

Problem 2: (10 points)

Let $\{f_n\}$ be a sequence of differentiable functions on a finite interval $[a,b]$ such that the functions themselves and their derivatives are uniformly bounded on $[a,b]$. Prove that $\{f_n\}$ has a uniformly converging subsequence.

Problem 3: (10 points)

Let $f \in L^1(\mathbb{R})$ and V_f be the closed subspace generated by the translates of f, i.e., $V_f := \{f(\cdot - y) \mid \forall y \in \mathbb{R}\}$. Suppose $\hat{f}(\xi_0) = 0$ for some ξ_0. Show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(\mathbb{R})$, then $\hat{f}$ never vanishes.

Problem 4: (10 points)

(a) State the Stone-Weierstrass theorem for a compact Hausdorff space X.

(b) Prove that the algebra generated by functions of the form $f(x,y) = g(x)h(y)$ where $g, h \in C(X)$ is dense in $C(X \times X)$.

Problem 5: (10 points)

For $r > 0$, define the dilation $d_r f : \mathbb{R} \to \mathbb{R}$ of a function $f : \mathbb{R} \to \mathbb{R}$ by $d_r f(x) = f(rx)$, and the dilation $d_r T$ of a distribution $T \in \mathscr{D}'(\mathbb{R})$ by

$$\langle d_r T, \phi \rangle = \frac{1}{r} \langle T, d_{1/r}\phi \rangle \qquad \text{for all test functions } \phi \in \mathscr{D}(\mathbb{R}).$$

(a) Show that the dilation of a regular distribution T_f, given by

$$\langle T_f, \phi \rangle = \int f(x)\phi(x)\, \mathrm{d}x,$$

agrees with the dilation of the corresponding function f.

(b) A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the δ-distribution is homogeneous of degree -1.

(c) If T is a homogeneous distribution of degree n, prove that the derivative T' is a homogeneous distribution of degree $n-1$.

Problem 6: (10 points)

Let $\ell^2(\mathbb{N})$ be the space of square-summable, real sequences $x = (x_1, x_2, x_3, \ldots)$ with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}.$$

Define $F : \ell^2(\mathbb{N}) \to \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} x_n^2 - x_n^4 \right\}$$

(a) Prove that F is differentiable at $x = 0$, with derivative $F'(0) : \ell^2(\mathbb{N}) \to \mathbb{R}$ equal to zero.

(b) Show that the second derivative of F at $x = 0$,

$$F''(0) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R},$$

is positive-definite, meaning that

$$F''(0)(h, h) > 0$$

for every nonzero $h \in \ell^2(\mathbb{N})$.

(c) Show that F does not attain a local minimum at $x = 0$.

Problem 7: (10 points)

Consider the dynamical system:

$$\begin{aligned} \dot{x} &= y^3 - 4x, \\ \dot{y} &= y^3 - y - 3x. \end{aligned}$$

Show that if a trajectory starts at any point on the line $x = y$, then it stays on it. Otherwise, show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ on any other trajectories.

Problem 8: (10 points)

Consider the system

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0,$$

where ϵ is a small parameter, and $h(x, \dot{x}) = (x^2 - 1)\dot{x}^3$. Show that a periodic orbit exists.

[Hint: Let $\langle \cdot \rangle$ be an averaging operator for a function defined over $[0, 2\pi]$, i.e., $\langle f \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta$. Calculate the averaged equation, $r' = \langle h(x, \dot{x}) \sin \rangle$ and identify its fixed points. You can use $\langle \cos^{2n+1} \sin^{2m+1} \rangle = 0$, $\langle \cos^{2n} \rangle = \langle \sin^{2n} \rangle = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$ to simplify your calculations.]