Graduate Group in Applied Mathematics
University of California, Davis

# Preliminary Exam

(September 27, 2001)

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**Problem 1.** (10 pts)
State the Riesz Representation Theorem for Hilbert spaces.

**Problem 2.** (10 pts)
Let $T$ denote a linear operator on the Hilbert space $\mathcal{H}$. What does it mean for $T$ to be *bounded*? Give an example of a bounded operator on $\mathcal{H} = L^2(-\infty, \infty)$ and give an example of an *unbounded* (linear) operator on the same space.

**Problem 3.** (10 pts)
Define a sequence of functions $p_n \in C([-1,1])$ by

$$\begin{aligned} p_{n+1}(x) &= \frac{1}{2}\left[p_n^2(x) + 1 - x^2\right] \qquad n = 0, 1, 2, \ldots, \\ p_0(x) &= 1. \end{aligned}$$

Show that $(p_n)$ is a monotone decreasing sequence of nonnegative polynomials. State Dini's theorem, and deduce that $(p_n)$ converges uniformly to the function $f : [-1,1] \to \mathbb{R}$ given by

$$f(x) = 1 - |x|.$$

**Problem 4.** (5 pts)
Let $X$ be a metric space with the property that every sequence $(x_n)$ such that

$$d\left(x^m, x^n\right) < 2^{-m} \qquad \text{for } n \geq m$$

is convergent. Prove that $X$ is complete.

**Problem 5.** (15 pts)
**(a)** (5 pts) Let $T$ denote a self-adjoint and compact (sometimes called completely continuous) operator on the Hilbert space $\mathcal{H}$. ($T$ need not be bounded.) State the spectral theorem as it applies to the operator $T$. Be sure to define all your symbols. (Think of this as a small essay.)
**(b)** (5 pts) Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Ku(x) = \int_0^1 k(x,y)u(y)\,dy, \qquad k(x,y) = \min(x,y).$$

Is this operator $K$ self-adjoint? Is $K$ also compact? Explain why or why not.
**(c)** (5 pts) Compute the eigenvalues of $K$ defined as above. Show that $K$ is a positive operator.

**Problem 6.** (15 pts)
**(a)** (5 pts) You may assume as given that the *oscillator wave functions*,

$$\varphi_k(x) = \frac{1}{\sqrt{2^k k!\,\sqrt{\pi}}}\exp(-x^2/2)H_k(x),\ k = 0,1,2,\ldots,$$

($H_k(x)$ are the Hermite polynomials) form a complete orthonormal system of the Hilbert space $L^2(-\infty,\infty)$. Explain why the identity

$$2 = \sum_{k=0}^{\infty}\left(\int_{-1}^{1}\varphi_k(x)\,dx\right)^2$$

must be satisfied. (Hint: The solution does not require the computation of difficult integrals!)
**(b)** (5 pts) For what values of $\varepsilon \in \mathbb{R}$ are the sequences

$$f = \left\{\frac{1}{n^\varepsilon}\right\}_{n=1}^{\infty}$$

elements of the Hilbert space $\ell^2$?
**(c)** (5 pts) For what values of $\varepsilon \in \mathbb{R}$ are the functions

$$f(x) = \begin{cases} 0 & : \quad x = 0 \\ 1/x^\varepsilon & : \quad 0 < x \le 1 \end{cases}$$

elements of the Hilbert space $L^2([0,1])$? If you change the value of $f$ at $x = 0$, can this change your answer? Explain why.

**Problem 7.** (15 pts)
**(a)** (5 pts) Define the potential energy and total energy for the system

$$x'' = -x^3 - x^2 + x + 1 \qquad (*)$$

**(b)** (5 pts) Plot the potential energy, being careful with the max/min points.
**(c)** (5 pts) Use the graph in (b) to construct a phase portrait for solutions of $(*)$. Justify your diagram.

**Problem 8.** (20 pts)
Consider the system

$$\begin{aligned} x' &= -x - xy^2 \\ y' &= -x^2 y \end{aligned}$$

**(a)** (10 pts) Linearize the system about rest point $(0,0)$, and determine the stability of this rest point for the linearized system.
**(b)** (10 pts) Prove that the rest point $(0,0)$ is *stable* for this nonlinear system.