Probability Preliminary Exam
(September 16th, 2025)

First name : _____	Student ID : _____
Last name : _____	Additional pages : _____

Instructions:

1. **All problems are worth 10 points.**
2. **Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**
3. **Use the front and back of each page to write the solution of each problem.**
4. **If you need extra pages, please do not use the same sheet for different problems.**
5. **Write your name and problem number on each additional page you use.**
6. **Combinatorial expressions such as Binomial coefficients need not be simplified.**
7. **Answers should be in closed form, not containing an integral or summation sign.**
8. **Leave answers in terms of the standard Normal c.d.f. Φ when appropriate.**

Equations Provided:

- Poisson(λ) — mean λ; variance λ; p.m.f. $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$
- Geometric(p) — mean $1/p$; possible values $k = 1, 2, 3, \ldots$
- Exponential(λ) — mean $1/\lambda$; p.d.f. $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$; c.d.f. $F(x) = 1 - e^{-\lambda x}$
- $N(0, 1)$ — p.d.f. $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)$; c.d.f. $\Phi(z)$

PROBLEM 1.

A pair (X, Y) of random variables has the joint density

$$f(x,y) = \begin{cases} c(x+y) & \text{if } x, y \in [0,1]; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c.

(b) Find the marginal densities of X and Y. Are X and Y independent?

(c) Find $\mathbf{P}(3X < Y \,|\, X < Y)$.

(d) Find $\mathbf{E}(XY)$.

PROBLEM 2.

A four-sided die has a 0, two 1's, and a 2 on its faces.

(a) Find the probability that the first three rolls show the same number.

(b) Find the probability that the first roll is strictly larger than the next five rolls.

(c) Let X be the sum of 200 rolls of this die. Using a relevant approximation, find the probability that $X = 200$. (Hint: use $\mathbf{P}(X = 200) = \mathbf{P}(199.5 \leq X \leq 200.5)$.)

(d) Let X be defined as in (c). Find an exact expression for the probability that $X = 200$. (Hint: note that the value of a single roll of this die has a well-known distribution. Use what you know about this distribution.)

PROBLEM 3.

Suppose that X and Y are independent and uniform on $[0, 1]$.

(a) Find $\mathbf{E}(e^X)$.

(b) Find $\mathbf{P}(|X - Y| \leq \frac{1}{2})$.

(c) Find the density f_W of the random variable $W = |X - Y|$.

PROBLEM 4.

Cards are dealt without replacement from a standard deck.

(a) Find the conditional probability that the second card is a club given that the first card is club.

(b) Find the conditional probability that the second card is a club given that the second-to-last card is a club.

(c) Find the conditional probability that the first card is a club given that none of the next five cards are clubs.

(d) Find the expected number of clubs that appear before the king of diamonds.

PROBLEM 5.

Roll a fair die repeatedly and let S_m be the sum of the numbers in the first m rolls.

(a) Find the probability that S_m ever equals 3.

(b) Estimate the probability that S_m ever equals 3000.

PROBLEM 6.

Consider a branching process where each individual has no children with probability $\frac{1}{4}$, one child with probability $\frac{1}{4}$, and two children with probability $\frac{1}{2}$.

(a) If the process is started with one individual, what is the probability that the process eventually dies out?

(b) If the process is started with k individuals, what is the probability that the process eventually dies out?

PROBLEM 7.

Let X_n be the following Markov chain on $\{0, 1, 2, \dots\}$. If the current state 0, the next state is

$$\begin{cases} 1 & \text{with probability } 1/3; \\ 0 & \text{with probability } 2/3. \end{cases}$$

If the current state is $i > 0$, the next state is

$$\begin{cases} i+1 & \text{with probability } 1/3; \\ i-1 & \text{with probability } 2/3. \end{cases}$$

(a) Find the stationary distribution.

(b) Start in state 0. Find the expected number of steps until the chain returns to 0 for the first time.

PROBLEM 8.

A fair coin is flipped repeatedly. Let N be the number of flips required to get two consecutive heads. Find $\mathbf{E}(N)$.