Applied Mathematics Preliminary Exam (June 17th, 2025)

First name : __________	Student ID : __________
Last name : __________	Additional pages : __________

Instructions:

1. **Unless indicated, all problems are worth 10 points.**
2. **Explain your answers clearly. Unclear answers will not receive credit.**
   State results and theorems you are using.
3. **Use the front and back of each page to write the solution of each problem.**
4. **If you need extra pages, please <u>do not</u> use the same sheet for different problems.**
5. **Write your name and the problem number in each additional page you use.**

PROBLEM 1.

Consider the following model for a chemical reaction

$$\frac{dc}{dt} = -rc + V\frac{c}{c+K} + I,$$

where c in the concentration of substance C in μM, t is time in sec, and r, V, and K are *positive* parameters with units sec^{-1}, $\mu M/sec$, and μM, respectively.

(a) Show that the model can be written in the dimensionless form

$$\frac{dx}{d\tau} = -\alpha x + \frac{x}{x+1} + \beta$$

for suitably defined dimensionless variable x and τ, and dimensionless parameters α and β. Verify the α and β are dimensionless.

(b) For the dimensionless model with $x > -1$, $\alpha > 0$, and $\beta = 0$,

(i) Find the steady states and determine their stabilities.

(ii) Sketch the bifurcation diagram using α as the bifurcation parameter (i.e., x vs α). Indicate the stability of the steady states on the diagram. At what values of α and x does a bifurcation occur? Classify the bifurcation.

(iii) Sketch the phase portraits the three qualitatively different cases.

(c) Now consider the general case for the dimensionless mode. Derive set of algebraic equation for the curves of bifurcations in α, β-parameter space. (DO NOT SOLVE THE EQUATIONS.)

PROBLEM 2.

Consider the system

$$\frac{dx}{dt} = x - y - x^3$$
$$\frac{dy}{dt} = x + y - y^3$$

Show that the system has a nontrivial stable periodic solution.

PROBLEM 3.

Consider the following inhomogeneous regular Sturm-Liouville system on the unit interval $[0, 1]$:

$$f" = g$$
$$f(0) = 0 = f'(1)$$

(a) (7 pts) Find the *Green's function* of this system.
(b) (3 pts) Suppose $g(x) = \sin(\pi x/2)$ Then, find the solution of the above system using the Green's function.

PROBLEM 4.

Consider a planar curve $y = y(x) \geq 0$, with $y(0) = y(1) = 0$, and rotate this curve about the x-axis to form a solid. Find the curve that *maximizes* the volume of such a solid under the constraint that its surface area is A where $0 < A < \pi$. Hint: The surface area of such a solid can be written as $\int_0^1 2\pi y\sqrt{1+(y')^2}\mathrm{d}x$.

PROBLEM 5.

Find the leading asymptotic solution for

$$\epsilon u'' + 2u' + u = 0, \quad u(0) = a, \quad u(1) = b, \quad \epsilon \ll 1,$$

where a and b are constants.

PROBLEM 6.

Find the leading asymptotic on the time scale $t = O(\epsilon^{-1})$ as $\epsilon \to 0$ for the nonlinear oscillator

$$\ddot{u} + u + \epsilon(u^2 - 1)\dot{u} = 0, \quad u(0) = a, \quad \dot{u}(0) = b, \quad \epsilon \ll 1$$