# Probability Preliminary Exam
# (June 2026)

**Instructions:**

1. **All problems are worth 10 points.**
2. **Explain your answers clearly. Unclear answers will not receive credit.**
   **State results and theorems you are using.**
3. **Use separate sheets for the solution of each problem.**
4. **Combinatorial expressions such as Binomial coefficients need not be simplified.**
5. **Answers should be in closed form, not containing an integral or summation sign.**
6. **Leave answers in terms of the standard Normal c.d.f. $\Phi$ when appropriate.**

**Equations Provided:**

- Poisson($\lambda$) — mean $\lambda$; variance $\lambda$; p.m.f. $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$
- Geometric($p$) — mean $1/p$; possible values $k = 1, 2, 3, \ldots$
- Exponential($\lambda$) — mean $1/\lambda$; p.d.f. $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$; c.d.f. $F(x) = 1 - e^{-\lambda x}$
- $N(0,1)$ — p.d.f. $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)$; c.d.f. $\Phi(z)$

**PROBLEM 1.**

A house has 7 rooms. Each room is randomly painted one of three colors: white, pink, or yellow.

(a) Find the probability of having exactly 4 white rooms.

(b) Find the conditional probability that all rooms are pink, given that no rooms are yellow.

(c) Find the conditional probability that exactly 3 rooms are pink, given that exactly 2 rooms are yellow.

(d) If there are no yellow rooms, the entire house is repainted randomly. This repeats until there is at least one yellow room. Find the probability that the final paint job has no pink rooms.

**PROBLEM 2.**

In a game of chance, you are given the choice between two options for the rules. If you pick the first option, you roll 100 dice and your score is the total sum of numbers rolled. If you pick the second option, you roll 1 die and multiply the result by 100 to get your score. In either case, you win if your score is 500 or more.

(a) Find the expectation and variance of your score under each option.

(b) Suppose you pick the first option. Use Chebyshev's inequality to give an upper bound on the probability of winning.

(c) Which option is better?

## PROBLEM 3.

Each chocolate bar contains a golden ticket with probability one in a million, i.e., probability $1/N$ where $N = 1,000,000$. Assume independence between chocolate bars.

(a) Open 10 chocolate bars. Find the (exact) probability of getting 3 or more tickets.

(b) Now open 5,000,000 chocolate bars. Using the relevant approximation, estimate the probability of getting 3 or more tickets.

(c) Open chocolate bars repeatedly until you get a ticket. Find the (exact) probability that you find the ticket in an even-numbered bar. (The first bar you open is $\#1$, the second is $\#2$, and so on.)

**PROBLEM 4.**

Assume that customers arrive at a store as a Poisson process of rate $\lambda$ per hour. Customers are either *difficult* or *easy*. Suppose that each customer that arrives is *difficult* with probability $\frac{1}{2}$, independently of all other customers.

(a) What is the expected time of the first arrival of a difficult customer?

(b) What is the expected waiting time until at least one customer of each type has arrived?

(c) Suppose that difficult customers stay in the store for two hours and easy customers stay in the store for one hour. Find the probability that there are exactly $k$ customers in the store at time $t = 5$.

## PROBLEM 5.

A Markov chain $X_n$ on $\{0, 1\}$ has the transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Start with $X_0 = 1$.

(a) Find $\lim_{n\to\infty} \mathbf{P}(X_n = 1)$.

(b) Write down an expression for $\mathbf{P}(X_4 = 0 \,|\, X_{10} = 1)$.

(c) Suppose that $W_n, Y_n$ and $Z_n$ are independent Markov chains all of which have transition matrix $P$. Let $S_n = W_n + Y_n + Z_n$. Then $S_n$ is a Markov chain on $\{0, 1, 2, 3\}$. (You are not expected to prove this.) Find the transition matrix of $S_n$.

(d) Find the stationary distribution of $S_n$. (Hint: this is easiest if you use the answer from part (a).)

## PROBLEM 6.

Consider a branching process where each individual has no children with probability $\frac{1}{4}$, one child with probability $\frac{1}{4}$, and two children with probability $\frac{1}{2}$.

(a) If the process is started with one individual, what is the probability that the process eventually dies out?

(b) If the process is started with $k$ individuals, what is the probability that the process eventually dies out?

(c) Suppose that we start with a random number of individuals $Y$. If $Y$ has generating function $G(s) := \mathbf{E}(s^Y) = e^{(s-1)}$, what is the probability that the process eventually dies out?