Numerical Analysis Preliminary Exam
(June 18th, 2025)

First name : __________________	Student ID : __________________
Last name : __________________	Additional pages : ______________

Instructions:

1. **All problems are worth 10 points.**
2. **Explain your answers clearly. Unclear answers will not receive credit.**
   State results and theorems you are using.
3. **USe the front and back of each page to write the solution of each problem.**
4. **If you need extra pages, please <u>do not</u> use the same sheet for different problems.**
5. **Write your name and problem number on each additional page you use.**

PROBLEM 1. Let $f(x) = x^2$, $x_0 = -1$, and $x_1 = 0$.

(a) Find the Lagrange interpolation polynomial of degree 1 using x_0 and x_1.

(b) Recall the following theorem:

Theorem. Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}([a, b])$. Then, for each $x \in [a, b]$, there exists $\xi(x) \in (a, b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

where $P(x)$ is the Lagrange interpolating polynomial to f at the points $x_0, x_1, \ldots, x_n$.

Use this theorem to find an upper bound for the approximation error of the polynomial interpolation in (a) for $x \in [-1, 0]$. We note that this upper bound need not be tight. Can you find a tight upper bound for $x \in [-1, 0]$?

(c) Find the quadratic Lagrange polynomial approximation to $f(x)$ using the interpolation points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. What is the approximation error in this case and why?

PROBLEM 2.

Let the two point quadrature formula on the interval $[-1, 1]$ use the quadrature nodes $x_1 = -\alpha$ and $x_2 = \alpha$, where $\alpha \in (0, 1]$:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx w_1 f(-\alpha) + w_2 f(\alpha).$$

(a) The formula is required to be exact whenever f is a polynomial of degree 1. Show that $w_1 = w_2 = 1$, independent of the value of α.

(b) Show that there is one particular value of α for which the formula is exact also for all polynomials of degree 2. Find this α.

(c) Show that for the value of α in (b), the formula is also exact for all polynomials of degree 3.

PROBLEM 3. Let $T \in \mathbb{R}^{n\times n}$ be tridiagonal:

$$T = \begin{bmatrix} a_1 & b_2 & & & \\ c_2 & a_2 & b_3 & & \\ & c_3 & a_3 & \ddots & \\ & & \ddots & \ddots & b_n \\ & & & c_n & a_n \end{bmatrix}.$$

(a) Suppose that the entries of T are such that $|a_1| > |b_2|$, $|a_n| > |c_n|$, and $|a_i| > |c_i| + |b_{i+1}|$ for $i = 2, 3, \ldots, n-1$. Show that T is invertible.

(b) Under the assumption on the entries in (a), show that T has an LU factorization. In other words, $T = LU$ where L is lower triangular and U is upper triangular.

(c) Show that the LU factorization in (b) may be computed in $O(n)$ operations.

(d) Suppose you have factored $T = LU$ as in the above step. Find an algorithm that takes as input $b \in \mathbb{R}^n$ and outputs the solution to $Tx = b$ in $O(n)$ operations.

PROBLEM 4. For any $v \in \mathbb{R}^n$ such that $\|v\|_2 = 1$, define $H_v = I - 2vv^T$.

(a) Show that H_v is symmetric for any v.

(b) Show that H_v is orthogonal for any v.

(c) Let $x \in \mathbb{R}^n$. Find a vector v with $\|v\|_2 = 1$ such that

$$H_v x = H \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

for some $\alpha \in \mathbb{R}$. What must $|\alpha|$ be equal to?

PROBLEM 5. Consider the scalar initial value problem (IVP)

$$\begin{aligned} y' &= f(t, y(t)), \quad t \in [0,1] \\ y(0) &= y_0. \end{aligned}$$

Let N be a positive integer, let $h = 1/N$, and $t_j = jh$ for $j = 0, 1, 2, \ldots, N$.

The leapfrog/midpoint method is a multistep method with time stepping specified by

$$y_{j+1} = y_{j-1} + 2hf(t_j, y_j), \quad j = 1, 2, \ldots, N-1.$$

(a) Find the rate at which the local truncation error goes to zero for the leapfrog method as $h \to 0$ when f is smooth.

(b) Show that the leapfrog method is zero stable.

(c) Find the region of absolute stability for the leapfrog method.

PROBLEM 6. Consider the initial value problem (IVP)

$$\begin{aligned} y' &= y^{1/5} \\ y(0) &= 0. \end{aligned}$$

(a) Show that

$$y(x) = \left(\frac{4x}{5}\right)^{5/4}$$

solves the IVP

(b) Show that the forward Euler method fails to approximate the solution to the IVP. Justify your answer.

(c) Now consider approximating the solution to the same initial value problem with the backward Euler method. Show that there is a solution of the form $y_n = (c_n h)^{5/4}$, for $n \geq 0$, with

$$\begin{aligned} c_0 &= 0 \\ c_1 &= 1 \\ c_n &> 1, \quad \text{for all } n \geq 2. \end{aligned}$$