Applied Mathematics Preliminary Exam (Fall 2015)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider the system

$$\dot{x} = -y - x^3,$$
$$\dot{y} = x^5.$$

(a) Is the equilibrium $(x, y) = (0, 0)$: (i) linearly stable; (ii) linearly asymptotically stable; (iii) hyperbolic? What do your answers imply about the nonlinear stability of the equilibrium?

(b) Find a Liapunov function for the system of the form

$$V(x, y) = Ax^6 + By^2.$$

What can you conclude about the nonlinear stability of $(0, 0)$ from the Liapunov function?

Problem 2 Consider the discrete dynamical system with iterates x_n given by the map

$$x_{n+1} = -\mu x_n - x_n^3,$$

where μ is a real parameter.

(a) Find the fixed points of the system as a function of μ and determine their linearized stability.

(b) What kind of bifurcation occurs at $x_n = 0$ as μ increases through $\mu = 1$?

(c) If x_n is small and $\mu = 1 + \epsilon$ is close to 1, show that

$$x_{n+2} \approx (1 + 2\epsilon)x_n + 2x_n^3,$$

after neglecting smaller terms. Determine whether the bifurcation in (b) is subcritical or supercritical.

Problem 3 Find among all continuous curves of length ℓ in the upper half-plane of $\mathbb{R}^2$ passing through $(-a, 0)$ and $(a, 0)$, the one that, together with the interval $[-a, a]$, encloses the largest area. Then, compute the maximum area too.

[Hint: You may want to use the *symmetry* of the problem to your advantage! Also, note that the length of the curve ℓ does not include the length of the interval $2a$ on the horizontal axis.]

Problem 4 Consider a simple rectangular domain $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < a, 0 < y < b\}$ with $a > b$, and the simple heat equation with the following initial and boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} & \text{for } (x, y, t) \in \Omega \times [0, \infty); \\ \dfrac{\partial u}{\partial x}(0, y, t) = \dfrac{\partial u}{\partial x}(a, y, t) = 0, & \text{on } 0 \le y \le b, t \in [0, \infty); \\ \dfrac{\partial u}{\partial y}(x, 0, t) = \dfrac{\partial u}{\partial y}(x, b, t) = 0, & \text{on } 0 \le x \le a, t \in [0, \infty); \\ u(x, y, 0) = f(x, y), & \text{on } (x, y) \in \Omega. \end{cases}$$

(a) Write down the general solution of this problem as a double Fourier series. [Hint: Use the separation of variables.]

(b) Identify the spatial modes (i.e., Fourier basis functions involving only (x, y) variables, not t) corresponding to the three lowest frequencies.

(c) Determine the solution of the above initial and boundary value problem in the case of $f(x, y) \equiv c =$ a real-valued constant.

Problem 5 Consider the following *regular* Sturm-Liouville problem (RSLP):

$$\begin{cases} f'' + \omega^2 f = g & 0 \le x \le 1; \\ f'(0) = 0 = f'(1), \end{cases}$$

where $\omega > 0$ is *not* an integer multiple of π.

(a) Find the Green's function for this RSLP.

(b) What happens if we try this with $\omega = 0$?

Problem 6 Find a one-term approximation, valid to order ε, of the solution to the following differential equation

$$\varepsilon \frac{d^2 y}{dx^2} + y\left(\frac{dy}{dx} + 3\right) = 0$$

for $0 < x < 1$, with boundary conditions $y(0) = -1$ and $y(1) = 1$.

It might be useful to know that

$$\int \frac{1}{-0.5x^2 + a} dx = \sqrt{\frac{2}{a}} \tanh^{-1}\left(x\sqrt{\frac{1}{2a}}\right) + b$$

where a is a positive constant and b is a constant.

It also might be useful to know that tanh is an odd function and that $\lim_{x\to\infty} \tanh(x) = 1$ and $\lim_{x\to-\infty} \tanh(x) = -1$.

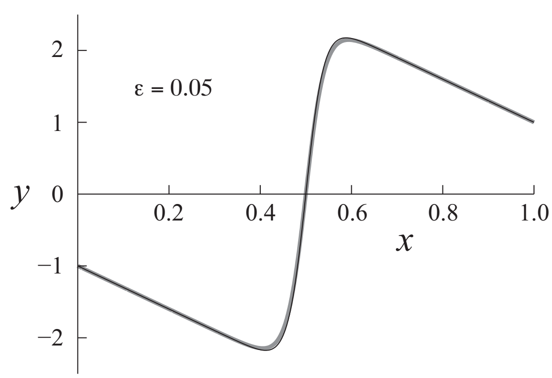

Figure 1: This is a numerical solution of the equation with $\varepsilon = 0.05$ (gray), plotted with my solution for the one-term approximation (black).