Spring 2013: PhD Applied Math Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1. Consider the 2×2 system of ODEs, where a, b are real constants,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} + a(x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix} + b(x^2+y^2)\begin{pmatrix} -y \\ x \end{pmatrix}.$$

(a) Linearize the system at the origin. Classify the equilibrium of the linearized system and determine its linearized stability.

(b) Write the system in polar coordinates, sketch the phase plane, and determine how the nonlinear stability of the origin depends on (a, b).

Problem 2. Consider the following initial-value problem for an infinite-dimensional system of ODEs for real-valued functions $\{x_1(t), x_2(t), x_3(t), \dots\}$

$$\frac{dx_n}{dt} = n^2 x_n^3, \qquad x_n(0) = c_n, \qquad n = 1, 2, 3, \dots.$$

(a) Solve for $x_n(t)$.

(b) If $\sum_{n=1}^{\infty} n^2 c_n^2 \leq 1$, show that a solution exists in some time interval $|t| < T$, and give an estimate for the minimal existence time $T > 0$.

(c) If $\sum_{n=1}^{\infty} c_n^2 \leq 1$, show that a solution need not exist in any interval $|t| < T$, however small one chooses $T > 0$,.

Problem 3. Let

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

where p, q are smooth, real-valued functions on on $a \leq x \leq b$ and $p(x) > 0$.

(a) Define the Green's function $G(x, \xi)$ for the regular Sturm-Liouville problem $Lu = f$ for $a < x < b$, with $u(a) = 0$, $u'(b) = 0$.

(b) Show that G is symmetric i.e. $G(x, \xi) = G(\xi, x)$, and give a physical interpretation of this symmetry.

Problem 4. Let $X = \{u \in C^2([1,2]) : u(1) = 0,\ u(2) = 1\}$ and define the functional $J : X \to \mathbb{R}$ by

$$J(u) = \int_1^2 \frac{\sqrt{1 + (u')^2}}{x}\, dx.$$

(a) Write down the Euler-Lagrange equation associated with J.

(b) Solve the Euler-Lagrange equation to find the minimizer of J on X.

Problem 5. Consider a vibrating string that is initially at rest and is subject to a spatially dependent, time-periodic external force with frequency ω. Suppose that the displacement $u(x,t)$ satisfies

$$\begin{aligned}
&u_{tt} - c^2 u_{xx} = A \sin\left(\frac{\pi x}{L}\right) \sin(\omega t) \qquad 0 < x < L,\ 0 < t,\\
&u(0,t) = 0, \qquad u(L,t) = 0, \qquad 0 \le t\\
&u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad 0 \le x \le L.
\end{aligned}$$

where $A \neq 0$ is the amplitude of the external force.

(a) Solve this IBVP for $u(x,t)$.

(b) For what values of ω is the solution also periodic in time?

Problem 6. Let $\nu > 0$ and $-\infty < U < \infty$ be constants, and consider the following PDE with boundary conditions at $x = \pm\infty$:

$$\begin{aligned}
&u_t + u u_x = \nu u_{xx}, \qquad -\infty < x < \infty,\ 0 < t,\\
&u(x,t) \to U \text{ as } x \to -\infty, \qquad u(x,t) \to 0 \text{ as } x \to \infty.
\end{aligned} \tag{1}$$

(a) If x, t, and u have dimensions of length, time, and velocity, respectively, show that this problem is dimensionally consistent. Determine the dimensions of ν and U, and use ν and U to nondimensionalize the problem.

(b) Consider traveling wave solutions $u = u(x - ct)$ of (1). What can you say using dimensional analysis about the speed c and a typical width L of a traveling wave?

(c) Find a first-order ODE for the traveling wave profile $u = u(z)$, where $z = x - ct$. Show that traveling waves exist if $U > 0$ but not if $U < 0$, and verify the results of the dimensional analysis.