Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
September 22, 2009

Instructions:

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

For $\epsilon > 0$, let η_ϵ denote the family of *standard* mollifiers on $\mathbb{R}^2$. Given $u \in L^2(\mathbb{R}^2)$, define the functions

$$u_\epsilon = \eta_\epsilon * u \text{ in } \mathbb{R}^2.$$

Prove that

$$\epsilon \|Du_\epsilon\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)},$$

where the constant C depends on the mollifying function, but not on u.

Problem 2: (10 points)

Let $B(0,1) \subset \mathbb{R}^3$ denote the unit ball $\{|x| < 1\}$. Prove that $\log|x| \in H^1(B(0,1))$.

Problem 3: (10 points)

Prove that the continuous functions of compact support are a dense subspace of $L^2(\mathbb{R}^d)$.

Problem 4: (10 points)

There are several senses in which a sequence of bounded operators $\{T_n\}$ can converge to a bounded operator T (in a Hilbert space $\mathcal{H}$). First, there is convergence in the norm, that is, $\|T_n - T\| \to 0$, as $n \to \infty$. Next, there is a weaker convergence, which happens to be called strong convergence, that requires that $T_n f \to Tf$, as $n \to \infty$, for every vector $f \in \mathcal{H}$. Finally, there is weak convergence that requires $(T_n f, g) \to (Tf, g)$ for every pair of vectors $f, g \in \mathcal{H}$.

(a) Show by examples that weak convergence does not imply strong convergence, nor does strong convergence imply convergence in norm.

(b) Show that for any bounded operator T there is a sequence $\{T_n\}$ of bounded operators of finite rank so that $T_n \to T$ strongly as $n \to \infty$.

Problem 5: (10 points)

Let $\mathcal{H}$ be a Hilbert space. Prove the following variants of the spectral theorem.

(a) If T_1 and T_2 are two linear symmetric and compact operators on $\mathcal{H}$ that commute (that is, $T_1 T_2 = T_2 T_1$), show that they can be diagonalized simultaneously. In other words, there exists an orthonormal basis for $\mathcal{H}$ which consists of eigenvectors for both T_1 and T_2.

(b) A linear operator on $\mathcal{H}$ is *normal* if $TT^* = T^* T$. Prove that if T is normal and compact, then T can be diagonalized.

(c) If U is unitary, and $U = \lambda I - T$, where T is compact, then U can be diagonalized.

Problem 6: (10 points)

Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.

Problem 7: (10 points)

Consider the equation

$$\frac{d^2x}{dt^2} + x - \epsilon x|x| = 0$$

(a) Find the equation for the conserved energy.

(b) Find the equilibrium points and the values of ϵ for which they exist.

(c) There are two qualitatively different phase portraits, for different values of ϵ. CLEARLY sketch and label these phase portraits.

(d) Show that there exist initial conditions, for any ϵ, for which solutions are periodic.

(e) For initial data $x(0) = a$, $\dot{x}(0) = 0$, calculate the first two terms (in ϵa) of the Taylor expansion of the period of the orbit in the limit $\epsilon a \to 0$.

Problem 8: (10 points)

Consider the system

$$\frac{dx}{dt} = ax + y - xf(x^2 + y^2)$$
$$\frac{dy}{dt} = -x + ay - yf(x^2 + y^2)$$

where a is real, f is continuous, $f(0) = 0$ and $f(z) \geq z^{1/2}$.

(a) Show that the origin is the only equilibrium point.

(b) Study the linear stability of the origin.

(c) Show that there exists a stable limit cycle if $a > 0$ (state and use the Poincaré-Bendixson theorem).

(d) Take the special case with $f(z^2) = z$ for all $z \geq 0$ with $a > 0$. Find the limit cycle explicitly by solving the system.