Applied Mathematics Preliminary Exam
(Spring 2019)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider a rigid, flat object, of mass m and length ℓ, attached to a torsional spring of stiffness κ in the presence of wind of speed v (see Fig. 1). The equations of motion are

$$\frac{m\ell^2}{4}\frac{d^2\theta}{dt^2} = -\kappa\theta + vc\frac{\ell}{2}\sin(\theta)$$

where c is a drag constant so that vc has units of force. Assume that all constants listed above are positive (but keep in mind that $\theta(t)$ may be negative).

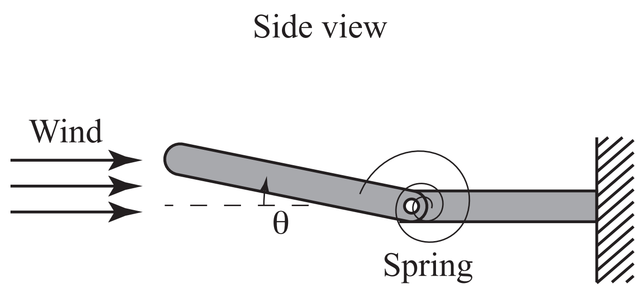

Figure 1:

(a) Use non-dimensionalization to show that the qualitative behavior of the system is defined by a single non-dimensional parameter.
(b) Show that there is a conserved quantity.
(c) As wind speed v increases from 0, find a critical value of the non-dimensional parameter at which a bifurcation occurs, and identify the type of bifurcation.
(d) Sketch the phase portrait at **i.** a wind speed just below the bifurcation and **ii.** a wind speed just after the bifurcation.

Problem 2 Consider the following predator-prey model:

$$\frac{dx}{dt} = x\big(x(1-x) - y\big) \qquad \frac{dy}{dt} = y(x-a)$$

where x is the (positive) non-dimensional population of prey, y is the (positive) non-dimensional population of predators, and a is a (positive) non-dimensional parameter.

(a) Sketch the null-clines in the first quadrant, $x, y \geq 0$.
(b) Find and classify all fixed points
(c) Find and classify all bifurcations that occur as a varies (assume $a > 0$).
(d) Show that a stable limit cycle exists for some values of a.

Problem 3 An *annular* plate with inner and outer radii $a < b$, respectively, is held at temperature B at its outer boundary and satisfies the boundary condition $\dfrac{\partial u}{\partial r} = A$ at its inner boundary, where A, B are constants. Find the temperature if it is at a *steady state*.

[Hint: It satisfies the two-dimensional Laplace equation and depends only on r. You can also use the fact that the Laplace operator can be expressed in the polar coordinate (r,θ) as: $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$.]

Problem 4 Let ω be positive, but *not* an integer multiple of π and consider the following boundary value problem on the unit interval $[0,1]$:

$$f'' + \omega^2 f = g, \quad f'(0) = 0 = f'(1).$$

(a) Find the *Green's function* for this boundary value problem.

(b) Discuss what happens if we try this with $\omega = 0$?

Problem 5 Let A be a symmetric matrix and let λ_0 be a simple (i.e. multiplicity one) eigenvalue of A with corresponding eigenvector $\mathbf{v}_0$. Derive an expression for the eigenvalue, λ, up to order ϵ in the limit of small ϵ to the problem

$$A\mathbf{v} + \epsilon \mathbf{F}(\mathbf{v}) = \lambda \mathbf{v}$$

that is λ_0 at leading order.

Problem 6 The van der Pol oscillator,

$$\epsilon \dot{u} = v + u - \frac{u^3}{3},$$
$$\dot{v} = -u,$$

exhibits periodic relaxation oscillations. The oscillation exhibits two time scales (a fast and slow time scale) for small ϵ.

Let $f(u) = u^3/3 - u$. The following information about f may be helpful:

$$f'(\pm 1) = 0$$
$$f(\pm 1) = \mp 2/3$$
$$f(\pm 2) = \pm 2/3$$

(a) Draw the nullclines in the phase plane (uv-plane), sketch the the limit cycle for small ϵ, and label the regions of fast and slow dynamics on the limit cycle.

(b) Compute the period of the oscillation at leading order as $\epsilon \to 0$.

End of the exam.