GGAM 207 Preliminary Exam (Spring 2014)

Instructions

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1 Consider

$$\begin{aligned}\frac{dx}{dt} &= x(a-x-y)\\ \frac{dy}{dt} &= (y-2a)(x-y)\end{aligned}$$

(a) Find all the equilibrium points.

(b) Find the linear (in)stability of each equilibrium point as a function of a.

(c) Sketch the phase portrait for representative values of a.

(d) Sketch the bifurcation diagram in the (a,x)-plane.

Problem 2 Find the shortest distance between two points (a,b) and (c,d) in $\mathbb{R}^2$ using the Calculus of Variations.

[Hint: Consider a curve $(x(t),y(t))$, $0 \le t \le 1$ with $(x(0),y(0)) = (a,b)$ and $(x(1),y(1)) = (c,d)$.]

Problem 3 Consider the following *regular* Sturm-Liouville problem:

$$\begin{cases}(xf')' + \lambda x^{-1} f = 0 & 1 \le x \le \mathrm{e};\\ f(1) = f(\mathrm{e}) = 0.\end{cases}$$

(a) Find the eigenvalues and *normalized* eigenfunctions of the above RSL problem. [Hint: Convert this into a simpler RSL problem using the change of variable of x.]

(b) Expand the function $g(x) \equiv 1$ in terms of these eigenfunctions.

Problem 4 Find the leading order uniform approximation to the solution $y(x)$ of

$$\epsilon y'' - (1+x)^2 y' + y = 0, \quad y(0) = 1, \quad y(1) = 0$$

in the limit $\epsilon \downarrow 0^+$. [Hint: boundary layer theory.]

Problem 5 Use the method of stationary phase to find the leading order approximation, as $x \to \infty$, of

$$\int_0^1 e^{ixt^2}\,dt.$$

Problem 6 A wave h of single frequency ω in a medium of variable speed $c(x) > 0$ satisfies

$$\frac{d}{dx}\left[c^2(x)\frac{dh}{dx}\right] + \omega^2 h = 0.$$

(a) What is the condition under which the WKB method produces a good approximation? Under this condition, compute the WKB approximation of h up to second order.

(b) Suppose $c(x) \to c_\pm$ as $x \to \pm\infty$. What are the wavelengths as $x \to \pm\infty$? Let $h_+ = \lim_{x\to\infty} |h(x)|$ and $h_- = \lim_{x\to-\infty} |h(x)|$. With the WKB approximation, determine h_+ in terms of $c_\pm$ and h_-.