Applied Mathematics Preliminary Exam
(Spring 2023)

Instructions:

1. **All problems are worth 10 points.**

2. **Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

3. **Use separate sheets for the solution of each problem.**

PROBLEM 1.

Consider the nonlinear system of ODEs

$$\begin{aligned} \frac{dx}{dt} &= -x + y - \mu \\ \frac{dy}{dt} &= e^x - y \end{aligned} \tag{1}$$

For $\underline{\mu = 1}$,

(a) Use linear stability analysis to assess the stability of the unique fixed point at (0,1). What conclusions can be made about the stability of the fixed point?

(b) Plot the nullclines of the system. Indicate the direction of the flow in the different regions of phase space. Sketch trajectories in the phase plane, and determine the stability of the fixed point at (0,1) for the nonlinear system?

Now, for $\underline{\text{general } \mu}$,

(c) Draw the bifurcation diagram for the system, and classify the bifurcation that occurs at $\mu = 1$.

PROBLEM 2.

(a) Consider the system

$$\ddot{x} + x^3 = 0.$$

Find a conserved quantity $E(x, \dot{x})$ for the system. Use $E(x, \dot{x})$ to help sketch solution trajectories in the $x, \dot{x}$-phase plane and classify fixed point at $(0, 0)$.

(b) Now add nonlinear damping to the system,

$$\ddot{x} + 3\dot{x}^3 + x^3 = 0.$$

Show that the system has a globally attracting fixed point at $(0, 0)$.

PROBLEM 3.

Define $\lambda : C^2(0,1) \to \mathbb{R}$ as

$$\lambda(v) = \frac{\int_0^1 v_x^2 dx - 2v(1)b + 2v(0)a}{\int_0^1 v^2 dx}, \quad \text{for} \quad v \in C^2(0,1).$$

Show that if $u = \underset{v \in C^2(0,1)}{\operatorname{argmin}} \lambda(v)$

$$-\Delta u(x) = \lambda(u)u(x), \quad x \in (0,1)$$

$$u_x(0) = a, \; u_x(1) = b.$$

PROBLEM 4.

Solve the initial boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx}, \; x \in [0,1], \; t \in [0,1] \\ u_t(x,0) &= 2\cos(\pi x) + 3\cos(4\pi x), \; x \in [0,1] \\ u(x,0) &= 0, \; x \in [0,1] \\ u_x(0,t) &= 0, \; t \in [0,1] \\ u_x(1,t) &= 0, \; t \in [0,1] \end{aligned}$$

PROBLEM 5.

Find the the first term in the asymptotic approximation of the integral

$$\int_0^\alpha e^{x\left[t\left(1-\frac{t^2}{\alpha^2}\right)\right]}\,dt.$$

for $\alpha > 0$, in the two limits (a) $x \to -\infty$ and (b) $x \to \infty$. In each case you will have to use a different method to approximate the integral.

PROBLEM 6.

Use boundary layer asymptotic analysis to find the leading order uniform solution of the boundary value problem

$$-\epsilon\frac{d^2y}{dx^2} + \left(a^2 + x^2\right)\frac{dy}{dx} - xy = 0$$

with boundary condition

$$y(0) = 1, \quad y(a) = 0.$$

for $0 < \epsilon \ll 1$. Sketch your solution, $y(x)$ clearly indicating the boundary layer and its thickness in powers of ϵ.