Applied Mathematics Preliminary Exam
(Spring 2016)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**
2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider the one-dimensional dynamical system

$$\frac{dx}{dt} = \mu x - 2x^2 + x^3$$

where $\mu \in \mathbb{R}$ is a parameter and $x(t) \in \mathbb{R}$.

(a) Determine the equilibria of the system and for what ranges of μ they exist.

(b) Determine the stability of the equilibria in (a).

(c) Sketch the bifurcation diagram for this system, using a solid line to denote a branch of stable equilibria and a dashed line to denote a branch of unstable equilibria. Classify the bifurcations that occur as μ increases from $-\infty$ to ∞.

Problem 2 (a) Show that the second order ODE

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + x = 0$$

can be put in the Hamiltonian form

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \tag{1}$$

by defining

$$p = e^{2x}\frac{dx}{dt}.$$

What is $H(x,p)$?

(b) Sketch the phase plane of the resulting Hamiltonian system (1).

Problem 3 Suppose a perfectly flexible rope of length $2a$ with uniform density ρ hangs under gravity from two fixed points $(-b,0)$ and $(b,0)$ in the xy-plane where $b<a$ and the gravity points downward (i.e., the negative y direction). Find the shape of this rope, $y=y(x)$, that minimizes the potential energy

$$V=\rho g\int_{-b}^{b} y\sqrt{1+(y')^2}\,\mathrm{d}x.$$

[Hint: The constraint is of course the arclength of the rope must be $2a$.]

Problem 4 Let $f(\theta)$ be the 2π-periodic function such that $f(\theta)=\mathrm{e}^{\theta}$ for $-\pi<\theta\le\pi$, and let $\sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta}$ be its Fourier series; thus $\mathrm{e}^{\theta}=\sum_{n=-\infty}^{\infty} c_n\mathrm{e}^{\mathrm{i}n\theta}$ for $|\theta|<\pi$.

(a) Compute c_n, $n\in\mathbb{Z}$, explicitly.

(b) If we formally differentiate this equation, we obtain $\mathrm{e}^{\theta}=\sum_{n=-\infty}^{\infty}\mathrm{i}nc_n\mathrm{e}^{\mathrm{i}n\theta}$. But then, $c_n=\mathrm{i}nc_n$ or $(1-\mathrm{i}n)c_n=0$, so $c_n=0$ for all n. This is obviously wrong; where is the mistake?

Problem 5 Find a one-term approximation, that is valid for long time scales, of the solution to the following differential equation

$$\varepsilon \frac{d^2 x}{dt^2} + \varepsilon \frac{dx}{dt} + x = \cos(t)$$

for $t > 0$, with initial conditions $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$.

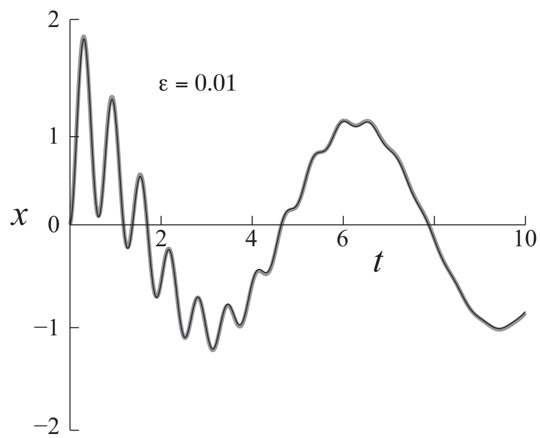

Figure 1: This is a numerical solution of the equation with $\varepsilon = 0.01$ (gray), plotted with my solution for the one term approximation, valid for long time scales (black).

Problem 6 Friedrichs' (1942) model problem for a boundary layer in a viscous fluid is

$$\varepsilon \frac{d^2 y}{dx^2} = a - \frac{dy}{dx}$$

for $0 < x < 1$ and $y(0) = 0, y(1) = 1$, and a is a given positive constant.

After finding the first term of the inner and outer expansions, derive a composite expansion for the solution to this problem.