Applied Mathematics Preliminary Exam
(Fall 2016)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 The SIR model is a simple and sometimes accurate way to describe the spread of a disease in a population. One variant of the model is given by the following three equations:

$$\begin{aligned}\frac{dS}{dt} &= a(I+R+S) - aS - bSI \\ \frac{dI}{dt} &= bSI - aI - cI \\ \frac{dR}{dt} &= cI - aR \end{aligned} \tag{1}$$

where S is the number of susceptible individuals, I the number of infected individuals and R the number of recovered individuals in the population and t is time.

The parameters are defined as follows:

a is the birth rate and also the death rate. Since these rates are equal, the population maintains a constant size, $R+I+S=N$, where N is a constant.

b is the transmission likelihood. When a susceptible and infected individual meet, the susceptible becomes infected with some probability. The parameter b defines the rate that susceptible and infected individuals meet and the infection is transmitted.

c is the recovery rate. An infected individual recovers at this rate, and then is immune to the disease.

a. Using a and N to define your time and population scales, respectively, non-dimensionalize the three differential equations.

Given the appropriate non-dimensionalization, and using the constraint that the population maintains a constant size, the equations become

$$\begin{aligned}\frac{dx}{dT} &= 1 - x - \alpha xy \\ \frac{dy}{dT} &= \alpha xy - (1+\beta)y \end{aligned} \tag{2}$$

where x is the probability that an individual is susceptible, y is the probability that an individual is infected, and the probability that an individual is resistant (z) can be determined from the constraint $x+y+z=1$.

b. Find all fixed points (x^*, y^*) and determine their stability for all combinations of $\alpha, \beta > 0$.

c. Suppose that $\beta = 1$. A bifurcation occurs as α changes. Classify this bifurcation, and sketch a phase portrait before and after the bifurcation.

Problem 2 Consider the following mechanical system.

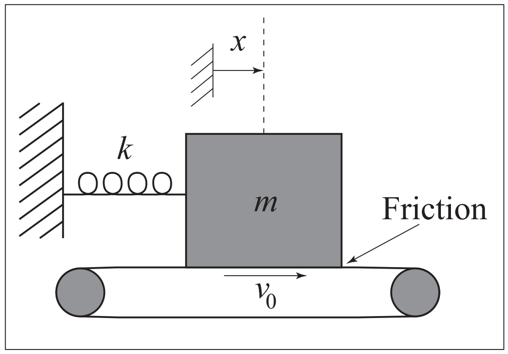

Figure 1: Mechanical system for problem 2.

A block, of mass m, sits on a conveyer belt moving at velocity v_0. The mass is attached to a wall with a linear spring of stiffness k. The position of the mass, x, as a function of time, t, obeys the following differential equation

$$m\frac{d^2x}{dt^2} = -kx - f(\dot{s})$$

where f is the frictional force that the conveyer belt applies to the block and $\dot{s}$ is the velocity of the block relative to the belt, $\dot{s} = \dfrac{dx}{dt} - v_0$. This equation can be non-dimensionalized to

$$\frac{d^2X}{dT^2} = -X - F\left(\frac{dX}{dT} - V\right) \tag{3}$$

Suppose that $V = 1$. Also, suppose that the friction force as a function of relative speed has the following form

$$F(x) = \begin{cases} 1 + ax & : \quad x > 0 \\ -1 + ax & : \quad x < 0 \end{cases} \tag{4}$$

a. Perhaps the simplest model of friction is Coulomb friction, which is Eq. 4 with $a = 0$. Show that linearization predicts that the unique fixed point, $X = -F(-1) = 1$, $dX/dT = 0$, is a center and explain why this is, in fact, a true center.

b. Show that, as a varies, the fixed point goes from a stable to an unstable spiral (assuming $|a| < 2$).

c. It turns out that when the fixed point becomes unstable, a limit cycle appears. This is a Hopf bifurcation. Is it a subcritical, supercritical or degenerate Hopf? Briefly (in a sentence or two) explain.

Problem 3 Define a functional $J : X \to \mathbb{R}$ by

$$J(u) = \int_0^{\pi/4} \left\{ \frac{1}{2}(u')^2 + \frac{1}{2}u^4 + u^2 \right\} dx$$
$$X = \left\{ u \in C^2([0,\pi/4]) : u(0) = 0,\ u(\pi/4) = 1 \right\}$$

a. What is the Euler–Lagrange equation for J?

b. Find the function $u \in X$ that minimizes J.

HINT: It turns out that $u'(0) = 1$, which may be helpful in evaluating the constants of integration.

Problem 4 Consider the boundary value problem (BVP)

$$u'' + u = f(x) \qquad 0 < x < 2\pi$$
$$u(0) = 0, \qquad u(2\pi) = 0,$$

for $u \in C^2([0,2\pi])$, where $f \in C([0,2\pi])$ is a given function.

a. Show that a necessary condition for the BVP to have a solution is that

$$\int_0^{2\pi} f(x) \sin x \, dx = 0.$$

b. If a solution of the BVP exists, show that there is a unique solution u such that

$$\int_0^{2\pi} u(x) \sin x \, dx = 0.$$

c. Write down the set of equations satisfied by the generalized Green's function $G(x,\xi)$ for this BVP. (You don't have to solve for G.)

d. Write down the BVP and orthogonality condition that are satisfied by the function

$$u(x) = \int_0^{2\pi} G(x,\xi) f(\xi) \, d\xi.$$

Problem 5 In the relativistic mechanics of planetary motion around the Sun, one comes across the problem

$$\frac{d^2u}{d\theta^2} + u = \alpha\left(1 + \epsilon u^2\right),$$

where $\alpha > 0$. Here, $u = 1/r$, where r is the normalized radial distance of the planet from the sun, and θ is the angular coordinate in the orbital plane. Find a first-term approximation of the solution u that is valid for large θ for small ϵ that satisfies the initial conditions

$$u(0) = 1$$
$$u'(0) = 0.$$

Problem 6 Find the leading order composite expansion for small ϵ for the problem

$$\epsilon^2 y'' + \epsilon\frac{3}{2}xy' - y = -x, \quad \text{for } 0 < x < 1$$
$$y(0) = 1,$$
$$y(1) = 2.$$