Applied Mathematics Preliminary Exam
(Spring 2018)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider the oscillator equation

$$\ddot{x} + F(x, \dot{x})\dot{x} + x = 0,$$

where $F(x, \dot{x}) < 0$ if $r \leq a$ and $F(x, \dot{x}) > 0$ if $r \geq b$ with $r^2 = x^2 + \dot{x}^2$ and $a < b$. Show that there is at least one closed orbit in the region $a < r < b$.

Problem 2 Consider the system of ordinary differential equations

$$\frac{dx}{dt} = x(x - 2y)$$
$$\frac{dy}{dt} = y(2x - y)$$

(a) Show that $(x, y) = (0, 0)$ is the unique fixed point of the system.

(b) Use linear stability analysis to classify the fixed point at $(x, y) = (0, 0)$. What can you conclude about the stability of $(0, 0)$ based on this analysis?

(c) Sketch the phase portrait of the system and describe the stability of $(x, y) = (0, 0)$.

Problem 3 Suppose you are given a string of length L. Suppose you arrange it to lie along a function, $f(x)$, where $f(0) = 0$. Of all possible potential arrangements of the string, which one maximizes the volume enclosed by it, V, when it is rotated about the x-axis?

DO NOT look for a closed form solution. Instead, leave your answer as a differential equation, boundary conditions, and a sufficient number of constraint equations to allow a clever person with a computer to find a solution.

Problem 4 A plucked string, fixed at both ends, obeys the differential equation

$$u_{tt} = c^2 u_{xx} - a u_t$$

with boundary conditions $u(0,t) = u(L,t) = 0$, and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$. In these equations, u is the local displacement of the string at position x, and c is a constant (t is time). When the constant a is zero, this is the wave equation; here, you will examine the effect of $a > 0$.

(a) Write the solution to the differential equation.

(b) What happens to the solution as $t \to \infty$?.

(c) Give a possible physical interpretation of the term $a u_t$.

Problem 5 Consider projectile motion with air resistance. The (dimensional) ODE for $x(t)$, the height of the object is

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} - \frac{k}{x+R}\frac{dx}{dt}, \tag{1}$$

where g is the gravitational constant, R is the radius of the earth, and k is a non-negative constant related to the air resistence. Suppose an object is launched from the surface ($x(0) = 0$) at a low velocity $\left.\frac{dx}{dt}\right|_{t=0} = v_0$ (with v_0 small).

1. Non-dimensionalize Equation (1) by finding appropriate re-scalings of x and t and define (two) small parameters in terms of your scaling choices. [HINT: Your choice of scaling should give the familiar physical problem valid when the initial velocity or displacement is much smaller than R and air resistance is negligible.]
2. Using the non-dimensionalized equations, find the leading order asymptotic expansion for the solution. [HINT: Your expansion should be in orders of the small parameter you defined above that is *independent* of the air-resistance parameter k.]
3. What equation would you need to solve to find the solution to the next highest order in the small parameter, include initial conditions, but DO NOT solve the equation.

Problem 6 Determine the first terms in the inner and outer expansions for the following boundary value problem:

$$\epsilon y'' - (2x+1)y' + 2y = 0$$

with $y(0) = 1, y(1) = 0$, and $\epsilon \ll 1$. Construct a first-order uniformly valid expansion for $y(x)$.

End of the exam.