Applied Mathematics Preliminary Exam
(Spring 2017)

Instructions:

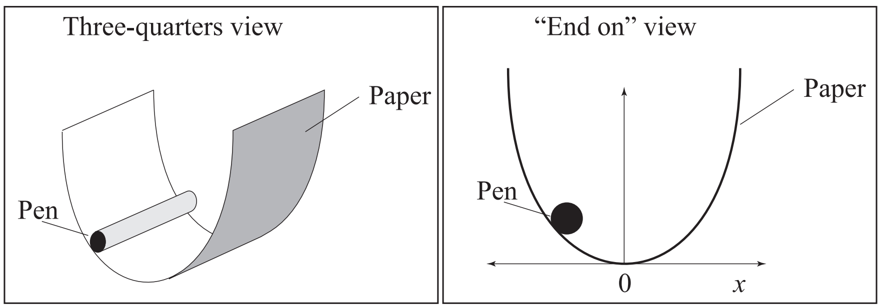
1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider the following system:

Step 1: Take a sheet of paper and hold it in a "U" shape.
Step 2: Place a pen or pencil near the bottom of the "U", but slightly off to one side.
Step 3: Let go, and watch the pen or pencil move back and forth.

Here's a sketch of the set-up.



When you do this experiment, the horizontal position of the pen (x in the sketch on the right) is a function of time t, and obeys the following equation (assuming conservation of energy)

$$\ddot{x} = \frac{-f'(x)}{a(1+f'(x))} - \frac{f''(x)}{2(1+f'(x))}(\dot{x})^2 \tag{1}$$

where $f(x)$ is a function that gives the height of the paper (in m) as a function of x (you may assume it and all of its derivatives are continuous), and a is a constant with units of s^2/m. Note that the dot indicates a time derivative and prime indicates a derivative with respect to x, which is measured in m.

a) Find all fixed points and determine their stability (your answer should depend on a, $f(x)$ and/or its derivatives). Note, for this question, use the Lyapunov definition of stability, where a fixed point is stable if trajectories that start sufficiently close (but not exactly at) the fixed point remain within some small neighborhood of the fixed point. You may assume that there is no point at which $f'(x) = f''(x) = 0$.

b) You might expect that the pen will oscillate about a stable fixed point. Find the period of this oscillation (your answer should depend on a, $f(x)$ and/or its derivatives). Your answer should include units.

Problem 2 Suppose you are studying the interaction of two proteins. The concentration of the first protein is $p(t)$ and the concentration of the second protein is $w(t)$. They interact via the following equations:

$$
\begin{aligned}
\dot{p} &= A_p \frac{p^2}{K_p^2 + p^2} - kwp \\
\dot{w} &= A_w \frac{w}{K_w + w} - kwp
\end{aligned}
$$

In each equation, the first term models the formation of protein, and the second term models the breakdown of the protein. Concentration is measured in units of number per liter and time is measured in seconds. The constants K_p and K_w then have units of concentration; the constants A_p and A_w have units of concentration per second; and k has units of inverse concentration per second.

a) Suppose that you know $K_p/K_w = \varepsilon$ (where ε is a small number), $A_w/(K_p^2 k) = \alpha/\varepsilon$ (where α is of order 1) and $A_p/(K_p^2 k) = \beta$ (where β is of order 1). Non-dimensionalize the equations and write them in terms of the appropriate non-dimensional variables and the non-dimensional constants ε, α and β.

b) Simplify the equations by expanding in ε and neglecting all terms of order ε.

c) Identify all fixed points and determine their stability. Discuss any bifurcations that may occur.

d) Sketch a phase portrait of the system. On your plot, be sure to (1) identify and classify all fixed points, (2) draw all null clines, (3) indicate the qualitative flow direction, and (4) sketch a few sample trajectories.

Problem 3 Let L be the differential operator

$$Lu = \left(1 + x^2\right) u'' - 2xu' \qquad 0 < x < 1$$

with boundary conditions $Bu = 0$ given by $u'(0) = 0,\ u'(1) = 0.$

(a) Find the adjoint L^* of L in $L^2(0,1)$ and the adjoint boundary conditions $B^* u = 0$.

(b) What are the solutions of the homogeneous boundary value problem $Lu = 0$, $Bu = 0$? What is the dimension of the null space?

(c) What are the solutions of the homogeneous adjoint boundary value problem $L^* u = 0$, $B^* u = 0$? What is the dimension of the null space?

Problem 4 (a) Let $\Omega = \{(x,y) \in \mathbb{R}^2 : -a < x < a,\ 0 < y < b\}$ be a rectangle. Use separation of variables to solve the boundary value problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \qquad (x,y) \in \Omega \\ &u(x,0) = 0, \qquad u(x,b) = e^{-x^2}, \\ &u_x(-a,y) = 0, \qquad u_x(a,y) = 0. \end{aligned}$$

(b) What is a physical interpretation of this problem? What is the approximate limiting behavior of the solution as $b \to 0$?

Problem 5

The equation for displacement, $q(x)$, of a nonlinear beam, on an elastic foundation and with an additional small forcing, is

$$q'''' - \kappa q'' + c^2 q = \epsilon \sin(\pi x), \quad \text{for } 0 < x < 1,$$
$$\kappa = \frac{1}{4}\int_0^1 (q_x)^2\, dx,$$
$$q(0) = q''(0) = q(1) = q''(1) = 0,$$

where c is a positive constant. Find a two-term expansion of the solution for small ϵ.

Problem 6 The dimensionless equation of motion of a frictionless pendulum is

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0.$$

In the limit of small amplitude (e.g. denote the amplitude of the θ as ϵ), the period is 2π to leading order. Compute the next term in the expansion of the period for small amplitude.

End of the exam.