Applied Mathematics Preliminary Exam
(Fall 2017)

Instructions:

1. **All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.**

2. **Use separate sheets for the solution of each problem.**

Problem 1 Consider the system of ordinary differential equations

$$\frac{dx}{dt} = x(1-x^2-y^2) - 2y(1+x)$$
$$\frac{dy}{dt} = y(1-x^2-y^2) + 2x(1+x).$$

(a) Use the function $V(x,y) = (1-x^2-y^2)^2$ like a Lyapunov function to prove the existence of an *asymptotically stable* closed orbit.

(b) Is the asymptotically stable closed orbit a limit cycle? Briefly justify your answer.

Problem 2 Consider the system of ordinary differential equations

$$\frac{dx}{dt} = x^2 - y$$
$$\frac{dy}{dt} = 2\alpha x - y - \beta$$

with the parameters $\alpha, \beta > 0$.

(a) Find and classify *all* bifurcations of steady states that occur in the system. (That is, identify all saddle-node, pitchfork, transcritical, and/or Hopf bifurcations. For any pitchfork or Hopf bifurcations, you do NOT have to determine whether they are super- or sub-critical).

(b) Plot the stability diagram (i.e., two-parameter bifurcation diagram) for the system in the α, β-plane. A codimension-2 bifurcation called a Taken-Bogdonov bifurcation occurs at $\alpha = 1/2, \beta = 1/4$. *Very briefly* describe what happens at this point.

Problem 3 Suppose you are given a string of length L. Of all possible potential arrangements of the string, which one maximizes the area enclosed by it, A?

Assume (1) that the shape of the string is symmetric; and
(2) that half of the string (of length $L/2$) can be described by the function $f(x)$, defined for $a \le x \le b$, such that $\int_a^b f(x)dx = A/2$.

Problem 4 A uniform, isotropic, linear-elastic beam of length L, subject to small transverse displacements has action $\mathscr{L}$,

$$\mathscr{L} = \int_0^L \left(-a^2 \frac{1}{2} u_x^2 + \frac{1}{2} u_t^2\right) dx,$$

where u is the local displacement at position x along the beam, and a is a constant (t is time, and the equation is non-dimensionalized).

(a) Derive a partial differential equation for the function, $u(x,t)$, that minimizes the action $\mathscr{L}$.

(b) Suppose that the beam is fixed at one end ($u(0,t) = 0$), and free at the other ($u_x(L,t) = 0$). Solve the PDE you derived in part (a) for arbitrary initial displacement ($u(x,0) = f(x)$) and zero initial velocity ($u_t(x,0) = 0$).

(c) Find the solution for the case where the beam is struck at the free end ($u_t(x,0) = b \cdot \delta(x - L)$, where b is an arbitrary positive constant, and $\delta(x)$ is the Dirac delta function).

Note, it may be useful to recall the property $\int_a^b f(x)\delta(x-c)\,dx = f(c)$, if $a < c < b$.

Problem 5 Use the WKB method to find an approximate solution to the following problem

$$\begin{cases} \varepsilon y'' + 2y' + 2y = 0 \\ y(0) = 0 \\ y(1) = 1 \end{cases}$$

HINT: Assume $y(x) = g(x)f(x)$ for some function $g(x)$ (that you must determine) to put the equation into the standard WKB form, namely $f''(x) - q(x)f(x) = 0$.

Problem 6 Assuming $\lambda \gg 1$, derive an approximation to the integral

$$I(\lambda) = \int_{-1}^{2} (1 + x^2) e^{-\lambda x^6} \, dx.$$

HINT: You may write your approximation in terms of the Gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx.$$

End of the exam.