Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(24 September 2003)

This exam has 3 numbered pages and is closed book. Please provide *complete* arguments. State or cite by name any major theorems you use. The Analysis portion of this exam is Problems 1-6. The ODE portion is Problems 7 & 8.

Problem 1. Recall the definition of the Gaussian distribution with variance $\sigma^2 > 0$:

$$p_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}), \ x \in \mathbb{R}.$$

For $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, define

$$(Tf)(x) = \sqrt{2} \int_{-\infty}^{+\infty} p_{1/2}(y) f(\sqrt{2}(x-y))\, dy.$$

a) Prove that p_1 is a fixed point of T.

b) Prove that for all $c > 0$, there is exactly one fixed point of T in $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, say f, such that $\|f\|_{L^1} = c$.

c) Let $g \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ be a non-negative function. Show that the sequence $T^n g$ converges in $L^1(\mathbb{R})$ as $n \to \infty$, and find its limit.

Problem 2. Let $(t_n)_{n\geq 1}$ be a sequence of non-negative real numbers such that $\sum_{n\geq 1} t_n^{3/2} = 1$. Let (a_n) be a sequence of complex numbers satisfying

$$\sum_{n\geq 1} |a_n|^3 < +\infty. \tag{1}$$

Define $f_n \in C([0,1])$, by

$$f_n(x) = \sum_{m=1}^{n} t_m a_m \sin(m\pi x)$$

Prove that the set

$$A = \{f_n \mid n \geq 1\}$$

is precompact in $C([0,1])$ with the supremum norm.

Problem 3.

a) Let X^{-1} be the distributional limit, as $\epsilon \to 0$, of the sequence of functions

$$F_\epsilon(x) = \begin{cases} \frac{1}{x}, & |x| > \epsilon \\ 0, & |x| < \epsilon \end{cases}$$

Show that X^{-1} is the distributional derivative of the function $f(x) = \log |x|$.

b) Show that the distributional limit, as $\epsilon \to 0$, of the following sequence

$$f_\epsilon(x) = \frac{1}{x - i\epsilon}, \qquad \epsilon > 0$$

is $X^{-1} + \pi i \delta$.

Problem 4. Let $h > 0$, and consider the following differential-difference initial-value problem, where $u(x,t)$ and $f(x)$ are 2π-periodic functions of x:

$$\begin{aligned} u_t(x,t) &= \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}, \\ u(x,0) &= f(x). \end{aligned}$$

a) (10 points) Use Fourier series to solve for $u(x,t)$ when $f(x)$ is square-integrable.

b) (5 points) How does the smoothness of $u(\cdot,t)$ for $t > 0$ compare with the smoothness of $f(\cdot)$?

c) (5 points) Discuss briefly what happens to your solution in the limit $h \to 0$.

Problem 5.

a) (5 points) Define "orthogonal projection on a Hilbert space".

b) (10 points) Suppose that P and Q are orthogonal projections with ranges $\mathcal{M}$ and $\mathcal{N}$, respectively. If $PQ = QP$, prove that $R = P + Q - PQ$ is an orthogonal projection. What is its range?

Problem 6.

a) (5 points) Define strong and weak convergence in a Hilbert space.

b) (5 points) Suppose that $(x_n)_{n=1}^{\infty}$ is an orthogonal sequence in a Hilbert space, meaning that x_n is orthogonal to x_m for $n \neq m$. Prove that the following statements are equivalent:

(i) $\sum_{n=1}^{\infty} x_n$ converges strongly;

(ii) $\sum_{n=1}^{\infty} x_n$ converges weakly;

$$\text{(iii)} \qquad \sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

c) (5 points) Give an example to show that if the sequence $(x_n)_{n=1}^{\infty}$ is not orthogonal, then $\sum_{n=1}^{\infty} x_n$ may converge weakly but not strongly.

Problem 7. Consider the system

$$\begin{aligned}\dot{q} &= 4p^3 - 4pq \\ \dot{p} &= 2p^2 - 3q^2\end{aligned}$$

a) Show that the function $H(q,p) = p^4 - 2p^2q + q^3$ is a conserved quantity for this system.

b) Compute the linearization of the system at the fixed point

$$(q^*, p^*) = \left(\frac{2}{3}, \sqrt{\frac{2}{3}}\right)$$

What type of fixed point is this? Sketch the behavior of the full system in a small neighborhood of the fixed point.

Problem 8. Consider the one-dimensional system

$$\dot{x} = x + \frac{rx}{1+x^2}$$

a) Compute the location of all fixed points as a function of $r \in \mathbb{R}$.

b) Plot the phase portrait when $r = -2$.

c) Plot a bifurcation diagram for the system. At what values of x and r does the bifurcation occur? What type of bifurcation is it?

d) Describe what would happen to the system's solution if it starts at $x = 1/2$ and $r = -2$, and then r is very slowly increased? Assume that the system dynamics are much faster than the change in r.