Graduate Group in Applied Mathematics
University of California, Davis
Preliminary Exam
January 4, 2008

Instructions:

- *This exam has 4 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

Problem 1: (10 points)

Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = (-1)^n x^n (1-x).$$

(a) Show that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $[0,1]$.

(b) Show that $\sum_{n=0}^{\infty} |f_n|$ converges pointwise on $[0,1]$ but not uniformly.

Problem 2: (10 points)

Consider $X = \mathbb{R}^2$ equipped with the Euclidean metric,

$$e(x,y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]^{1/2},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} e(x,y) & \text{if } x, y \text{ lie on the same ray through the origin,} \\ e(x,0) + e(0,y) & \text{otherwise.} \end{cases}$$

Here, we say that x, y lie on the same ray through the origin if $x = \lambda y$ for some positive real number $\lambda > 0$.

(a) Prove that (X, d) is a metric space.

(b) Give an example of a set that is open in (X, d) but not open in (X, e).

Problem 3: (10 points)

Suppose that $\mathscr{M}$ is a (nonzero) closed linear subspace of a Hilbert space $\mathscr{H}$ and $\phi : \mathscr{M} \to \mathbb{C}$ is a bounded linear functional on $\mathscr{M}$. Prove that there is a unique extension of ϕ to a bounded linear function on $\mathscr{H}$ with the same norm.

Problem 4: (10 points)

Suppose that $A : \mathscr{H} \to \mathscr{H}$ is a bounded linear operator on a (complex) Hilbert space $\mathscr{H}$ with spectrum $\sigma(A) \subset \mathbb{C}$ and resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$. For $\mu \in \rho(A)$, let

$$R(\mu, A) = (\mu I - A)^{-1}$$

denote the resolvent operator of A.

(a) If $\mu \in \rho(A)$ and

$$|\nu - \mu| < \frac{1}{\|R(\mu, A)\|},$$

prove that $\nu \in \rho(A)$ and

$$R(\nu, A) = \left[I - (\mu - \nu) R(\mu, A)\right]^{-1} R(\mu, A).$$

(b) If $\mu \in \rho(A)$, prove that

$$\|R(\mu, A)\| \geq \frac{1}{d\left(\mu, \sigma(A)\right)}$$

where

$$d\left(\mu, \sigma(A)\right) = \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is the distance of μ from the spectrum of A.

Problem 5: (10 points)

Let $1 \leq p < \infty$ and let $I = (-1, 1)$ denote the open interval in $\mathbb{R}$. Find the values of α as a function of p for which the function $|x|^\alpha \in W^{1,p}(I)$.

Problem 6: (10 points)

Let $\Omega = \{x \in \mathbb{R}^3 \ : \ |x| < 1\}$ denote the unit ball in $\mathbb{R}^3$. Suppose that the sequences $\{f_k\}$ in $W^{1,4}(\Omega)$ and that $\{\vec{g}_k\}$ in $W^{1,4}(\Omega; \mathbb{R}^3)$. Suppose also that there exist functions $f \in W^{1,4}(\Omega)$ and $\vec{g}$ in $W^{1,4}(\Omega; \mathbb{R}^3)$, such that we have the weak convergence

$$f_k \rightharpoonup f \text{ in } W^{1,4}(\Omega),$$
$$\vec{g}_k \rightharpoonup \vec{g} \text{ in } W^{1,4}(\Omega; \mathbb{R}^3).$$

Show that there are subsequences $\{f_{k_j}\}$ and $\{\vec{g}_{k_j}\}$ such that we have the weak convergence

$$\vec{D} f_{k_j} \cdot \operatorname{curl} \vec{g}_{k_j} \rightharpoonup \vec{D} f \cdot \operatorname{curl} \vec{g} \quad \text{in} \quad H^{-1}(\Omega).$$

Notation. Here f is a scalar function and $\vec{g} = (g_1, g_2, g_3)$ are three-dimensional vector-valued function. $\vec{D}$ denotes the three-dimensional gradient $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\text{curl}\,\vec{g} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \times \vec{g}$

As customary, we use $H^{-1}(\Omega)$ to denote the dual space of the Hilbert space $H_0^1(\Omega)$ consisting of those functions in $H^1(\Omega)$ which vanish on the boundary (in the sense of trace). Two useful identities are that

$$\begin{aligned} \text{curl}\,(\vec{D}f) &= 0 \quad \text{for any scalar function } f, \\ \text{div}\,(\text{curl}\,\vec{w}) &= 0 \quad \text{for any vector function } \vec{w}, \end{aligned}$$

where $\text{div}\,\vec{F} = \partial_{x_1} F_1 + \partial_{x_2} F_2 + \partial_{x_3} F_3$ denotes the usual divergence of a vector field $\vec{F} = (F_1, F_2, F_3)$.

Hint. Test $\vec{D}f_{k_j} \cdot \text{curl}\,\vec{g}_{k_j}$ with a function $\psi \in H_0^1(\Omega)$ and use integration by parts to argue the weak convergence.

Problem 7: (14 points)

The rotating bead on a hoop is a Hamiltonian system where

$$H(\theta, \Omega) = \frac{\Omega^2}{2} - \left(\frac{g}{R} \cos(\theta) - \frac{\omega^2}{4} \cos(2\theta) \right)$$

where θ is the angle that the bead makes from the vertical measured from "straight down," Ω is the angular velocity of the bead, ω is the angular velocity of the hoop, g is the acceleration of gravity, and R is the radius of the hoop. Recall that the Hamiltonian is conserved (it is the total energy) and the dynamics are given by

$$\dot{\theta} = \frac{\partial H}{\partial \Omega}, \quad \dot{\Omega} = -\frac{\partial H}{\partial \theta}.$$

(a) Write down the dynamical system.

(b) When the hoop is not rotating, this is exactly equivalent to the classical pendulum. Non-dimensionalize this system using a natural time scale associated with the classical pendulum. This will leave you with one parameter, call it λ which we shall use to study bifurcations.

(c) Find the value of λ_c at which a bifurcation occurs.

(d) Sketch the phase portrait for λ greater than the bifurcation value and less than the bifurcation value.

(e) Find the fixed points and classify their stability.

(f) Find the frequency of oscillation about either of the two neutrally stable fixed points for $\lambda > \lambda_c$.

(g) Sketch the phase portrait for $\lambda > \lambda_c$ if we add a damping term to the equation, i.e., $\dot{\Omega} = -\nu\Omega$ with $\nu > 0$.

Problem 8: (6 points)

Estimate the period of the limit cycle in the system

$$\ddot{x} + k\left(x^2 - 4\right)\dot{x} + x = 1$$

for $k \gg 1$. There are different ways to do this. One way to start involves recognizing the Lienard transformation, i.e. first write the system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x} + k\left(\frac{x^3}{3} - 4x\right)\right] + x = 1.$$

Second, define the quantity in square brackets to be $k\,y$. Third, write down the dynamical system for $\dot{x}$ and $\dot{y}$. From here you can find an approximate expression for the limit cycle and integrate the resulting equation to estimate the period.