Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(25 September 2002)

This exam has 3 pages (9 problems) and is closed book.

Problem 1. (10 points) Prove that R^1 with each of the metrics

(i) $\rho(x,y) = |\arctan(x) - \arctan(y)|$

(ii) $\rho(x,y) = |\exp(x) - \exp(y)|$

is incomplete and find its completion in each case.

Problem 2. (10 points) Let $\{c_k\}_{k=-\infty}^{+\infty}$ be the Fourier coefficients of an integrable function $f \in L^1(T^1)$ on a unit circle. Find the Fourier coefficients of the Steklov function

$$f_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(y)dy.$$

What can be said about their behavior as $h \to 0$?

Problem 3. (5 points) Prove or disprove that $C[0,1]$ with the usual sup norm is a Hilbert space.

Problem 4. (10 points) Consider a sequence of functions $f_n(x) = \frac{1}{n^{10}+1} \times \exp(-nx^2)$ in the Schwartz space $S(R^1)$. Prove or disprove that it converges to zero (in the Schwartz space topology) as $n \to \infty$.

Problem 5. (10 points) Let λ be an eigenvalue of the Fourier transform on R^1.

(i) Prove that the absolute value of λ is one.

(ii) Prove that $\lambda^4 = 1$.

Problem 6. (10 points) Consider a convolution with $f(x) = \frac{sin(\pi x)}{\pi x}$ as a linear operator on $L^2(R^1)$. Prove that it is a self-adjoint operator and find its norm.

Problem 7. (10 points) Let $\{e_k\}_{k=1}^{\infty}$ be a natural orthonormal basis in $l^2(Z_+^1)$. Define a sequence of linear operators in $l^2(Z_+^1)$ by the formula

$$A_n e_k = \delta(n, k) e_1, \quad k = 1, 2, \dots, \ n = 1, 2, \dots,$$

where $\delta(n, k)$ is the Kronecker symbol (i.e. it is equal to one when $n = k$ and it is equal to zero otherwise). Prove that

(i) $\|A_n\| = 1$.

(ii) A_n strongly converges to zero as $n \to \infty$ (i.e. for any vector x one has $A_n x \to 0$.)

Problem 8. For each of the equations

$$(i) \ x' = rx - 4x^3, \quad (ii) \ x' = r^2 - x^2$$

answer the following questions.

a) (5 points) Determine the bifurcation point of r and sketch the different types of vector field, including the fixed point(s), for r smaller than, equal to and bigger than the bifurcation point.

b) (5 points) Sketch the bifurcation diagram (i.e. the fixed point(s) versus r) and indicate the stability of the fixed point(s) on the diagram.

Problem 9. Consider the equation

$$x'' + \mu(x^2 - 1)x' + x = 0$$

and answer the following questions.

a) (5 points) Reduce the second order equation to a system of 1-st order equations by introducing a new variable.

b) (5 points) For the equilibrium point $x = 0, x' = 0$ find a Lyapunov function (i.e. a function which is 0 at the equilibrium but otherwise strictly positive or negative in a neighborhood of the equilibrium and changes its value monotonically along any trajectory in that neighborhood).

c) (5 points) Determine the stability of the equilibrium point (your answer should depend on μ). At what value of μ does the equilibrium change its stability?

d) (5 points) What is the range of μ for which the system has a stable limit cycle?