# Graduate Group in Applied Mathematics
# University of California, Davis
# **Preliminary Exam**
# March 29, 2012

**Instructions:**

- *This exam has 3 pages (8 problems) and is closed book.*
- *The first 6 problems cover Analysis and the last 2 problems cover ODEs.*
- *All problems are worth 10 points.*
- *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- *Use separate sheets for the solution of each problem.*

**Problem 1:** (10 points)
For $u \in L^1(0,\infty)$, consider the integral

$$v(x) = \int_0^\infty \frac{u(y)}{x+y}\,\mathrm{d}y$$

defined for $x > 0$. Show that $v(x)$ is infinitely differentiable away from the origin. Prove that $v' \in L^1(\epsilon,\infty)$ for any $\epsilon > 0$. Explain what happens in the limit as $\epsilon \to 0$.

**Problem 2:** (10 points)
Let $X \subset L^2(0,2\pi)$ be the set of all functions $u(x)$ such that

$$u(x) = \lim_{K\to\infty} \sum_{k=-K}^{K} a_k e^{ikx} \text{ in } L^2\text{-norm, with } |a_k| \le (1+|k|)^{-1}.$$

Prove that $X$ is compact in $L^2(0,2\pi)$.

**Problem 3:** (10 points)
For $\epsilon > 0$, we set

$$\eta_\epsilon(x) = \frac{1}{\pi} \sin\left(\frac{\epsilon \pi x}{x^2+\epsilon^2}\right) \frac{\epsilon}{x^2+\epsilon^2},$$

and define the convolution for $u \in L^2(\mathbb{R})$:

$$\eta_\epsilon * u(x) = \int_{\mathbb{R}} \eta_\epsilon(x-y) u(y) \, \mathrm{d}y.$$

For $\epsilon > 0$, prove that $\sqrt{\epsilon}(\eta_\epsilon * u)(x)$ is bounded as a function of $x$ and $\epsilon$, and that $\eta_\epsilon * u$ converges strongly in $L^2(\mathbb{R})$ as $\epsilon \to 0$. What is the limit?

**Problem 4:** (10 points)
Let $u_n : [0,1] \to [0,\infty)$ denote a sequence of measurable functions satisfying

$$\sup_n \int_0^1 u_n(x) \log(2 + u_n(x)) \, \mathrm{d}x < \infty.$$

If $u_n(x) \to u(x)$ almost everywhere, show that $u \in L^1(0,1)$ and that $u_n \to u$ in $L^1$ strongly.

(**Hint.** One possible strategy is Egoroff's Theorem.)

**Problem 5:** (10 points)
Let $u : [0,1] \to \mathbb{R}$ be absolutely continuous, satisfy $u(0) = 0$, and

$$\int_0^1 |u'(x)|^2 \, \mathrm{d}x < \infty.$$

Prove that

$$\lim_{x \to 0^+} \frac{u(x)}{x^{\frac{1}{2}}}$$

exists and determine the value of this limit.

**Problem 6:** (10 points)
Consider on $\mathbb{R}^2$ the distribution defined by the locally integrable function

$$E(x,t) = \begin{cases} \frac{1}{2} & \text{if } t - |x| > 0 \\ 0 & \text{if } t - |x| < 0 \end{cases}.$$

Compute the distributional derivative

$$\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2}.$$

**Problem 7:** (10 points)

Consider

$$\dot{x} = y + ax(1 - 2b - x^2 - y^2) \qquad \dot{y} = -x + ay(1 - x^2 - y^2)$$

with $0 < a \leq 1, 0 \leq b < \frac{1}{2}$; prove that there is at least one limit cycle and calculate the period $T(a,b)$ (*i.e.*, write it as an integral).

**Problem 8:** (10 points)

Consider the system

$$\dot{x} = y - 2x \qquad \dot{y} = \mu + x^2 - y\,.$$

(a) Sketch the nullclines of the system for different values of $\mu$ in order to find and classify the bifurcation that occurs at $\mu = \mu_c$.

(b) Classify the fixed points and sketch the phase portrait for $\mu$ slightly smaller than $\mu_c$.

(c) For which values of $\mu$ the system admits a stable spiral?