## PROBLEM 1.

Consider the conservative system

$$\frac{d^2x}{dt^2} = -x^2 + (r-2)x + (r-1) \tag{1}$$

where $r$ is a parameter.

(a) Find a conserved quantity (the energy function) $E(x, y)$ for the system, where $y = \frac{dx}{dt}$.

(b) Explicitly show that $E(x, y)$ is constant along trajectories, i.e., show that

$$\frac{d}{dt}(E(x(t), y(t))) = 0.$$

What does this imply about the relationship between the level sets of $E(x, y)$ and the solutions to the differential equation $(x(t), y(t))$?

(c) Find all fixed points of the system and determine their linear stabilities? What is their *nonlinear* stability in the nonlinear system?

(d) Sketch the bifurcation diagram for the system. At what value of $r = r_c$ is there a bifurcation? What type of bifurcation is it? *Briefly* justify your answer.

(e) Sketch trajectories in the $x, y$-phase plane for $r < r_c$, $r = r_c$, and $r > r_c$ .

(When drawing the phase plane for $r = r_c$, it will likely help to think about the type of fixed points that coalesce at this bifurcation point. Based on this, what is an appropriate name for the fixed point at this bifurcation point?)

**PROBLEM 2. Do problem 2(a) <u>OR</u> problem 2(b).**

(a) Consider a gradient system

$$\frac{d\vec{x}}{dt} = -\nabla V,$$

where $\vec{x} = (x, y)$ and $V(\vec{x})$ is a continuously differentiable, real-valued scalar function.

Suppose that $V(\vec{x}) > 0$ for all $\vec{x} \neq \vec{x}^* = (x^*, y^*)$ and $V(\vec{x}^*) = 0$. Show that the system has a globally asymptotically stable fixed point at $\vec{x}^*$.

(b) Consider a gradient system

$$\begin{aligned} \frac{dx}{dt} &= -2x - y^2 \\ \frac{dy}{dt} &= -2xy - 4y^3 \end{aligned} \quad . \tag{2}$$

Find the potential function $V(x, y)$ for the system and use it as a Lyapunov function to show that $(0, 0)$ is a globally asymptotically stable fixed point.

## PROBLEM 3.

Consider the eigenvalue problem

$$\begin{aligned}&Au + \lambda u = 0, \qquad A = x\frac{d}{dx}\left(x\frac{d}{dx}\right), \quad 1 \le x \le e,\\ &u(1) = u(e) = 0\end{aligned} \tag{3}$$

for the operator $A$ where $\lambda$ is the eigenvalues to be determined.

(a) Let

$$\langle u, v\rangle_w = \int_1^e \overline{u}(x)v(x)w(x)dx$$

be the inner product with respect to which $A$ is symmetric. Determine the weight function $w > 0$.

(b) Find the eigenvalues and eigenfunctions (e.g. by making a change of variable of $x$ to simplify the equation).

**PROBLEM 4.**

Consider

$$Lu := \frac{d}{dx}\left(x\frac{d}{dx}\right)u + \frac{\lambda}{x}u, \qquad x \in (1,e),$$

where $\lambda \in \mathbb{R}$ is a given parameter.

(a) Suppose $Lu = 0$ is the Euler-Lagrange equation for the Lagrangian

$$J[u] := \int_1^e p(u')^2 + qu^2 dx$$

with the boundary condition $u(1) = u(e) = 0$. Determine $p$ and $q$.

(b) For $\lambda = 0$ find the Green function for $L$ with the boundary conditions

$$u(1) = 0, \quad \frac{du}{dx}(e) = 0.$$

## PROBLEM 5.

Consider the second order ODE which describes the height, $h(x)$, of a wave in a medium with varying wave speed. The local wavenumber $k(x)$, depends on location $x$

$$\frac{d}{dx}\left[\frac{1}{k(x)^2}\frac{dh}{dx}\right] + h = 0. \tag{4}$$

In the example we will work with

$$k(x) = k_- + k_+ \frac{(\tanh(x/L) + 1)}{2}. \tag{5}$$

where $0 < k_- < k_+$.

(a) Non-dimensionalize this system using $L$ to measure length.

(b) What is the condition that WKB theory will yield good approximation of this equation? Introduce a small parameter, $\epsilon$, which describes the separation of scales associated with this condition.

(c) Assuming that WKB theory is valid for the given $k(x)$, find the approximation of $h(x)$ by carrying out the WKB expansion to the first two non-trivial orders of the theory. Do this by considering a general form of $k(x)$ - you need not substitute the particular $k(x)$ from equation (5).

(e) For the wave speed given in (5), if the wave profile is

$$h(x) = Ae^{ik_- x} \quad \text{for} \quad x \to -\infty$$

determine

$$|h(x)| \quad \text{for} \quad x \to \infty.$$

## PROBLEM 6.

Consider the boundary value problem

$$\epsilon y'' + 2y' - y = 0 \quad \text{on} \quad x \in [0,1], \quad 0 < \epsilon \ll 1$$

with

$$y(0) = 1, \quad y(1) = 0.$$

Using the method of boundary layer asymptotics, find the first order inner, outer, and uniform solution to this problem. To get full credit, ensure that you explain the idea of an overlap region.