# Analysis Preliminary Exam

## Applied Mathematics, September, 2004

**Write solutions on the paper provided, putting each problem on a separate page. Justify all of your mathematics. Print your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.**

**(1)** Consider the first order ordinary differential equation

$$\dot{x} = x^3 - 3x^2 + x, \tag{1}$$

where $x = x(t)$ is a real valued function of real variable $t$. Recall that a subset $S$ of $\mathbf{R}$ is said to be a *positively invariant region* for (1) if, whenever $x(0) \in S$, then $x(t) \in S$ for all $t \geq 0$. Prove that $[0, 4] = \{x : 0 \leq x \leq 4\}$ is a positively invariant region for (1).

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**(2)** Consider the second order ordinary differential equation

$$\ddot{x} + x^3 = 0. \tag{2}$$

Prove that (2) has no unbounded solutions.

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**(3)** Consider the following equation for an unknown function $f : [0,1] \to \mathbf{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x - y)^2 f(y) dy + \frac{1}{2} sin(f(x)). \tag{3}$$

Prove that there exists a $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all $g \in C([0,1])$, (3) has a unique continuous solution.

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**(4)** Let $\mathbf{X}$ be a normed linear space, and let $\mathbf{X}^*$ be its dual.

**(a)** State the Hahn-Banach Theorem for $\mathbf{X}$.

**(b)** Use the Hahn-Banach Theorem to prove that if the pair $x, y \in \mathbf{X}$ has the property that $\phi(x) = \phi(y)$ for all $\phi \in \mathbf{X}^*$, then $x = y$.

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**(5)** Suppose that $f \in H^1([a,b])$ and $f(a) = f(b) = 0$. Prove the *Poincare Inequality*

$$\int_a^b |f(x)|^2 dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(x)|^2 dx. \tag{4}$$

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**(6)** Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$.

**(a)** Give the definitions of the *point*, *continuous* and *residual* spectrum of $A$.

**(b)** A complex number $\lambda$ is said to belong to the *approximate spectrum* of $A$ if there is a sequence $(x_n)$ of vectors in $\mathcal{H}$ such that $||x_n|| = 1$, and $(A - \lambda I)x_n \to 0$ as $n \to \infty$. Prove that the appoximate spectrum is contained within the spectrum, and contains the point and continuous spectrum.

**(c)** Give an example to show that a point in the residual spectrum need not belong to the approximate spectrum.

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**(7)** Let $f_n : [0, +\infty) \to \mathbf{R}$ be a sequence of continuous, $n = 1, 2, 3...$, and suppose that there exists a constant $c \geq 0$ such that $f_n(x) \geq -c$ for all $x \geq 0$, $n \geq 1$. Prove that, for all $n \geq 1$, we have

$$\int_0^\infty e^{-f_n(x) - x} dx \geq e^{\int_0^\infty e^{-x} f_n(x) dx}. \tag{5}$$

(Hint: $d\mu = e^{-x} dx$ is a probability measure.)