Winter 2007: Applied Math Preliminary Exam
Part I: Analysis

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $S : C([0,1]) \to C([0,1])$ be a bounded linear operator. Suppose that $\|S(p)\| \leq 2$ for all polynomials p. Show that S is the zero operator.

Problem 2. For $p \geq 1$, let $l^p(\mathbb{N})$ be the set of sequences (x_n) such that

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

(a) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \subseteq l^q(\mathbb{N})$.

(b) Show that if $1 \leq p < q < \infty$ then $l^p(\mathbb{N}) \neq l^q(\mathbb{N})$.

Problem 3. Suppose that for some function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y);$$

in particular, both limits exist. Does it follow that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exists?

Problem 4. Let X be a metric space. A function $f : X \to X$ is said to be a contraction if there exists a $C < 1$ such that $d(f(x), f(y)) < Cd(x,y)$ for all $x \neq y$. The function f is said to be a *weak contraction* if $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$, without the constant C. The contraction mapping theorem says that if f is a contraction, then it has a fixed point. Show that the theorem also holds when f is a weak contraction and X is compact.

Problem 5. Construct the Green's function for the Dirichlet boundary-value problem

$$-u'' + 4u = f, \quad u(0) = u(2) = 0.$$

Problem 6. Let U be a unitary operator on a Hilbert space. Prove that the spectrum of U lies on the unit circle.

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Part II: ODE Theory

1. Show that the system

$$\begin{aligned}\dot{x} &= -x + y^3 - y^4 \\ \dot{y} &= -2x - y + 2xy\end{aligned}$$

has no periodic solutions. What is the asymptotic behavior, as $t \to \infty$, of the trajectory starting at $(\pi, -e^2)$? (Hint: Choose a, m and n such that $V = x^m + ay^n$ is a Liapunov function.)

2. Consider the system

$$\ddot{x} = -x^2 + (r-2)\,x + r - 1,$$

where r is a parameter.

(a) Show that there is a bifurcation at $r = r_c$ for some r_c. Find the value of r_c. What kind of bifurcation is it?

(b) Classify the fixed points of this nonlinear system. Give your reasons.