Graduate Group in Applied Mathematics
University of California, Davis

Preliminary Exam

(3 January 2003)

This exam has 2 numbered pages and is closed book. The Analysis portion of this exam is Problems 1-6. The ODE portion is Problems 7-9.

Problem 1. Let $f(x)$ be a real-valued function from $L^2(R^1)$ and

$$\alpha = \int_{-\infty}^{+\infty} f(x)\exp(-x^2)dx, \quad \beta = \int_{-\infty}^{+\infty} f(x)x\exp(-x^2)dx.$$

a) (5 points) Prove that $\alpha^2 < \pi \int_{-\infty}^{+\infty} f^2(x)dx$.

b) (5 points) Prove that $\alpha^2 + 2\beta^2 < \pi \int_{-\infty}^{+\infty} f^2(x)dx$.

Problem 2. Calculate the Fourier coefficients of the functions $f(x)$ and $g(x)$ in $L^2(0, 2\pi)$ where

a) (5 points) $f(x) = \cos^6(x)$,

b) (5 points) $g(x) = x - \pi$.

Problem 3.

a) (5 points) Prove or disprove that R^n equipped with the usual Euclidean norm is separable (i.e. it has a countable dense subset). Does the answer depend on the particular choice of norm in R^n?

b) (5 points) Prove that $l^\infty(\mathbb{Z})$ with the usual sup norm is not a separable space.

Problem 4. (10 points) Find all non-negative integers n and m such that $x^n \frac{d^m\delta(x)}{dx^m}$ is identically zero, where $\delta(x)$ is the delta function.

Problem 5. Consider the Hilbert space $L^2[-1, 1]$.

a) (5 points) Find the orthogonal complement of the space of all polynomials. Hint: Use the Stone-Weierstrass theorem.

b) (5 points) Find the orthogonal complement of the space of polynomials in x^2.

Problem 6. (15 points) Consider the space of all polynomials on $[0, 1]$ vanishing at the origin with the sup norm. Prove that the space is not complete and find its completion.

Problem 7. Consider the 2-d system

$$x' = x, \quad y' = -y + x^2.$$

a) (5 points) Show that the system has a saddle point at (0,0) and its stable manifold is the y-axis.

b) (5 points) Let (x, y) be a point on the unstable manifold and close to (0,0). Write the $y = u(x)$ and assume

$$u(x) = \sum_{k \geq 1} c_k x^k.$$

Determine the coefficients c_k (and thus $u(x)$) by substituting the expression into the equations.

c) (5 points) Check that your analytical result produces a curve with the same shape as the stable manifold shown in the figure.

Problem 8. (10 points) Show that $x' = y, y' = -x - x^3$ has a fixed point at the origin that is a center (i.e. the Jacobian has purely imaginary eigenvalues). Are the trajectories in a small neighborhood of the origin closed (i.e. periodic orbits)? Prove your answer.

Problem 9. (10 points) Sketch an argument for the existence of a periodic orbit for the system

$$x'' + x + \epsilon(x^2 - 1)x' = 0$$

with a small positive parameter ϵ.