PROBLEM 1. Consider the equation

$$\frac{dx}{dt} = rx + ax^2 - x^3,$$

where $-\infty < r < \infty$, $-\infty < a < \infty$ are parameters.

(a) For each a, there is a bifurcation diagram x vs r. Sketch the qualitatively different bifurcation diagrams that can be obtained by varying a, i.e., draw bifurcation diagrams for $a < 0$, $a = 0$, and $a > 0$.

(b) Summarize your results by plotting the regions in (r, a) parameter space that correspond to the qualitatively different classes of vector fields. Bifurcations occur on boundaries of these regions; identify the types of bifurcation that occur.

PROBLEM 2. Consider the system of equations

$$\frac{dx}{dt} = \lambda x - y - xr^2 + \lambda\frac{x^3}{r}$$

$$\frac{dy}{dt} = x + \lambda y - yr^2 + \lambda\frac{x^2 y}{r}$$

where r is the polar radius, i.e., $r^2 = x^2 + y^2$. Show that this system has a stable limit cycle when $\lambda > 0$.

PROBLEM 3. Let

$$\mathcal{E}[y(x)] = \int_0^1 \left[\frac{d^2y}{dx^2}\right]^2 dx.$$

Derive the equation for $y(x)$ which extremizes $\mathcal{E}$ subject to the constraint

$$\int_0^1 [y(x)]^2\, dx = 1$$

and the boundary conditions $y(0) = y(1) = y'(0) = y'(1) = 0$, where $y' = \frac{dy}{dx}$. Find the functions which extremize $\mathcal{E}$. What would be considered the natural boundary conditions for this problem?

PROBLEM 4. Find the solution $G(x, \xi)$ (the Green's Function) to the problem

$$\frac{d^2G}{dx^2} - G = \delta(x - \xi)$$

with $G(0, \xi) = G(L, \xi) = 0$.

Use the Green's function to compute the solution of

$$\frac{d^2y}{dx^2} - y = H(x)$$

with $y(0) = y(L) = 0$, and

$$H(x) = \begin{cases} 0 & 0 \le x \le \frac{L}{2} \\ 1 & \frac{L}{2} < x \le L \end{cases}$$

PROBLEM 5. A simple harmonic oscillator is subject to weak nonlinear damping proportional to the square of its velocity, and its displacement $y(t;\epsilon)$ satisfies the nondimensionalized initial value problem

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt}\left|\frac{dy}{dt}\right| + y = 0, \qquad y(0;\epsilon) = 1, \quad \frac{dy}{dt}(0;\epsilon) = 0.$$

Use the method of multiple scales to find a leading order approximation for $y(t;\epsilon)$ as $\epsilon \to 0^+$, valid on times $t = O(\epsilon^{-1})$. HINT. The function $\sin t|\sin t|$ has the Fourier expansion

$$\sin t|\sin t| = \sum_{n=1}^{\infty} b_n \sin nt, \qquad b_1 = \frac{8}{3\pi}.$$

PROBLEM 6. Use the method of matched asymptotic expansions to find a leading order composite approximation for the solution $y(x;\epsilon)$ of the following boundary value problem on $0 \le x \le 1$:

$$\epsilon y'' + 2y' + y^3 = 0, \qquad y(0;\epsilon) = 0, \quad y(1;\epsilon) = \frac{1}{2}.$$